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DESIGN OF GUIDANCE AND CONTROL SYSTEMS FOR OPTIMUM UTILIZATION OF INFORMATION

by L. Meier, J. Peschon, B. Ho, R. Larson, and R. Dressler

Prepared by
STANFORD RESEARCH INSTITUTE
Menlo Park, Calif.
for Ames Research Center

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ABSTRACT

A comprehensive scheme is described for designing control systems in the presence of uncertainties about initial plant state and plant parameters, disturbance inputs, and measurement noises. The method is based upon treating these random effects as perturbations on a nominal trajectory corresponding to plant operation without the random effects and with a nominal control as input. As a first approximation, the deterministic optimum is used for the nominal control, with optimal linearized estimation and control of the plant about the nominal trajectory. Next, the sensitivity of system performance to the random effects is computed; if this sensitivity is too great, then improved performance may be obtained by use of adaptive control (*i.e.*, estimation of the uncertain parameters) and modification of the nominal control. If adaptive control is not used, reduced sensitivity can also be obtained by modifying the linear estimation and control in addition to the nominal control. Performance computations are accurate to terms of second degree in the perturbations.

Optimization of the measurement system from two points of view is considered: First, a method is developed for the optimum choice of instruments which trades off increased cost against improved performance. Second, the optimum measurement subsystem control policy is developed for situations in which the measurement system may be operated in more than one mode.

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CONTENTS

CONTENTS

CONTENTS

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ILLUSTRATIONS

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TABLES

I INTRODUCTION

This final report summarizes the work performed on Contract NAS 2-3476 from 17 March, 1966 to 17 April, 1967. Three quarterly reports[1,2,3]* containing partial results obtained in the course of the study have been published previously.

Contract NAS 2-3476 is a logical extension of work performed by the same authors from November 1964 to September 1967 under Contract NAS 2-2457 entitled "Information Requirements for Guidance and Control Systems,"[4,5,6]

A. Objectives

The objective of Contract NAS 2-2457 was to relate the performance of a guidance or control system to the information-handling characteristics of its key components, notably the measurement subsystem. It was shown that the desired relations could be derived within the mathematical framework of combined optimum control and estimation theory.[5,6]

The objective of Contract NAS 2-3476 has been to proceed beyond the analysis of the effects of imperfect information upon system performance and to synthesize systems in which the degrading effect of imperfect information is minimized.

Specifically, the following statement of objectives was agreed upon:[7]

(1) To provide NASA with practical approaches toward the design and evaluation of systems in which optimum, or near optimum, utilization of information is necessary.

(2) To further the state of the art of the information and control sciences by providing mathematical relations between the relevant variables (such as performance, measurement subsystem outputs, control subsystem outputs, etc.) under general circumstances; by interpreting the physical significance of these relations; and by describing special cases that allow practical application and implementation in the near future.

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* References are listed at the end of the report.

B. Summary of the Results Obtained

In the course of Contract NAS 2-3476, the results discussed below were obtained:

1. Establishment of a Design Methodology

A methodology for designing guidance and control systems in the presence of uncertainty has been established and is described in Sec. II. The procedure starts with an optimum deterministic design, in which the existence of uncertainty is at first ignored. Thereafter, the sensitivity of the performance of this deterministic design with respect to the uncertainties is established. Finally, the optimum control signal originally found for the deterministic design is corrected to minimize the degradation of performance caused by the uncertainties. Several types of corrections, ranging from simple closed-loop control to the incorporation of the dual and stochastic effects are discussed.

2. Development of Sensitivity Equations

Sensitivity equations have been developed relating system performance to the various uncertainties, notably plant noise, measurement noise, and inaccurately known plant parameters. It has been shown in the course of the study that the desired sensitivity relations can be obtained under certain assumptions by second-order Taylor-series expansion of the Hamilton-Jacobi equation. With this sensitivity information, the degrading effect upon performance of noise and parameter uncertainty can be analyzed methodically. These relations not only permit a quantitative analysis of the effects of uncertainty but, in addition, suggest synthesis procedures for designing systems in which the degrading effect of uncertainty is minimized, that is, systems that utilize the available information optimally. The analysis and synthesis procedures derived from second-order sensitivity theory are presented in Sec. III and a very versatile computer program implementing these procedures is discussed in Part 3 of Appendix B.

3. Development of a Methodology for the Design of Adaptive Systems

The purpose of adaptive systems is to reduce the degrading effect of parameter uncertainty upon system performance. This is customarily accomplished by first estimating the imperfectly known parameters and thereafter changing the law of control to account for the present best estimate of these uncertain parameters. Using the sensitivity relations established in the course of the contract, it is straightforward to obtain an upper bound for the performance improvement made possible by an adaptive design. If this improvement is sufficient to justify the additional complexity of an adaptive system, then the synthesis procedures discussed in Part 2 above can be applied to achieve an optimum adaptive design. This approach toward adaptive system design is discussed in Sec. III. In the course of Contract NAS 2-2457, it was shown that the "dual control effect" arises in systems with imperfect state information. This effect dictates that the optimal control should accomplish the dual purpose of forcing the desired plant motion and of acquiring improved state information. The adaptive design procedure derived in Sec. III displays this dual control effect by shifting the nominal trajectory away from the deterministic optimum trajectory.

4. Development of an Optimum Design Procedure for Instrumentation Systems

The control system designer usually is not only responsible for the design of the law of control, but also for the selection and specification of the instrumentation subsystem. Ideally, he would like to measure every state variable with perfect accuracy; practically, this is rarely possible because of the excessive cost (or bulk) of the resulting instrumentation system. He must therefore relax the specifications of the instrumentation system until they fit his budget. How this can be done with minimum degradation in system performance is discussed in Sec. IV.

5. Development of Systems with Optimum Control of the Instrumentation Subsystem

The possibility of improving system performance by controlling the instrumentation subsystem as well as the plant was first explored by the authors under Contract NAS 12-59 for NASA Electronics Research Center, and was further developed under the present contract. This work constitutes the analytical basis for various proposed instrumentation systems with a built-in capability for adapting their sensors (dynamic range, quantization grain, etc.) to the measured data. A paper entitled "Optimum Control of Measurement Systems" has been accepted for presentation at the 1967 Joint Automatic Control Conference and publication in the *Transactions on Automatic Control* and is reproduced in Sec. V.

6. Study of Potential Applications

Contracts NAS 2-2457 and NAS 2-3476 were not aimed at any specific applications, and the results obtained are perfectly general. These results furthermore have reached a sufficient degree of perfection to be applicable to practical control and guidance problems. Generally speaking, they first allow the designer to analyze the performance degradation caused by uncertainty and next allow him to design a control system capable of coping in the best possible manner with these uncertainties. In order to find specific applications for these techniques, presently used and projected electronic systems for commercial and VSTOL aircraft were reviewed. The general results of this study are contained in Sec. VI.

Although the techniques developed in the course of the study were aimed primarily at systems under complete computer control, it is believed that the fundamental concepts used can be extended to certain design features of control systems containing man in the loop. In particular, these techniques provide an estimate of the performance achievable when the human operator is given incomplete information; the logical next step in the procedure determines which information must be made available if a prescribed level of performance is to be achieved and thus specifies the nature and characteristics of the display systems required.

A preliminary discussion on how these techniques might be used to design a human interface is contained in Sec. VI.

C. Unsolved Problems

The objectives pursued in Contracts NAS 2-2457 and NAS 2-3476 were to establish analysis and synthesis procedures for guidance and control systems required to operate in the presence of uncertainty. Generally speaking, these objectives were achieved in a practically acceptable manner. The approach requires that the deviations from the nominal caused by uncertainty be sufficiently small and that the appropriate functions can, in fact, be expanded into a Taylor series. These conditions appear to be satisfied by a majority of guidance and control systems operating on physical processes. Hard saturation of the control can be included in the theory by use of penalty functions.

To determine those classes of problems that cannot be handled, we must look for large perturbations and mathematical models that do not allow Taylor series expansions. Large perturbations arise, notably in conjunction with component failures, whereas mathematical models that cannot be linearized are characteristic of discrete-state/discrete-control systems. These models are not frequently encountered in the control of physical processes, but are very common in operations research and the management sciences.

With the exceptions quoted above, the methods developed apply to numerous practical control problems and provide most of the answers desired by the control engineers, except those pertaining to the implementation of the data-processing subsystem or controller.

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II A METHODOLOGY FOR DESIGNING GUIDANCE AND CONTROL SYSTEMS REQUIRED TO OPERATE IN THE PRESENCE OF UNCERTAINTY

A. Introduction

In Sec. I, we listed various analytical techniques that were developed in the course of Contracts NAS 2-2457 and NAS 2-3476 to assist the designer of control and guidance systems required to operate in the presence of uncertainty. In this section, we establish a methodology, based on these techniques, to guide the designer of such systems through a practical step-by-step procedure. This methodology is not the only possible approach; however, it constitutes a reasonable compromise between excessive complexity and unreal simplicity. Since the rigorous treatment of systems perturbed by random influences naturally leads to a combined optimization problem, the solution of which is difficult or impossible, it is easy for the designer to get bogged down by excessive complexity. The over simplified approach neglects these random influences altogether to solve a deterministic problem; the answers obtained in this manner can easily be meaningless, depending upon the effect of the random influences.

The methodology given below also constitutes a reasonable compromise in terms of the classes of systems it covers: Inclusion of all possible classes of systems (*e.g.*, finite state systems, systems with multiple controllers, etc.) would make the treatment of the vast majority of continuous guidance and control systems unnecessarily cumbersome.

B. System Description and Ground Rules

Since we are concerned with the tasks assigned to the control engineer and not with those assigned to the administrator or the mechanical engineer, we assume that

(1) The performance function (or, equivalently, the cost function) has been specified by the project administrator

(2) The plant to be controlled has already been designed by the mechanical, electrical, chemical, etc., engineer.

If the control system is not capable of achieving the prescribed performance with the given plant, a new overall design iteration may need to be carried out.

To summarize, we assume then that the control system designer has been given the cost function J, which in the general variational case takes on the customary form

$$J = \int_0^T l(x,u,t)dt + \Phi(x_T, T) \quad , \tag{II-1}$$

subject to constraining equations of motion of the form

$$\dot{x} = f(x,u,w,p,t) \quad , \tag{II-2}$$

where

x = state vector

u = control or decision vector

w = perturbation vector

p = parameter

$[0,T]$ = optimization interval.

In addition, there may be inequality constraints on x and u of the form

$$x \in X \qquad u \in U \quad . \tag{II-3}$$

Although the variational optimization problem represented by Eqs. (II-1) and (II-2) describes the majority of control and guidance problems in a very satisfactory manner, the static optimization problem discussed in greater detail in Appendix A is also frequently encountered. Since this is a simpler problem and yet displays most of the effects of interest in our discussion, it will be used liberally for illustrative purposes. This problem is briefly formulated as follows: Given the cost function

$$J = F(x,u) \quad , \tag{II-4}$$

it is desired to minimize J subject to a set of equality constraints of the form

$$g(x,u,w,p) = 0 \quad , \tag{II-5}$$

and a possible set of inequality constraints of the form

$$u \in U \quad \text{and} \quad x \in X \quad . \tag{II-6}$$

In Eq. (II-4) the vector u represents the independent (control) variables and the vector x represents the dependent variables; thus the dimension of g is taken equal to the dimension of x. As before, w and p represent perturbation and parameter vectors.

For both the variational and static problem formulation, the task assigned to the control engineer consists of minimizing the cost function subject to the plant constraints (II-2) and (II-4), respectively, but also subject to additional practical constraints relating to the complexity of the proposed control system, the major constituents of which are the instrumentation subsystem and the data processing subsystem (controller).

To complete the description of the plant, it remains necessary to discuss the significance of the variables x, u, p and w. The dependent or state variables x are not necessarily obvious and their selection is influenced by the accuracy requirements of the model. For example, the dynamic effects characterized by a short time-constant compared to the dominant time-constant of the plant may be neglected, in which case the dimension of the state vector is reduced, but the optimum control $u(t)$ corresponding to this simplified model may not be adequate for the real plant. Similarly, the control vector u is not always fixed *a priori*; for convenience, the designer may wish to maintain some of the input signals to the plant at a constant level and treat them as part of the parameter vector p; alternatively, he may connect these inputs to the controller, in which case they become part of the vector u. The vectors p and w enter in an identical manner into Eqs. (II-2) and (II-4) and it is convenient (but certainly not necessary) to differentiate between them. In what follows, p is used to denote constant parameters of which the designer's knowledge is uncertain. The vector w, on the other hand, denotes random perturbations, the future variation of which cannot be predicted with certainty. To summarize, p is taken to be a random variable; w is a random process.

For the case of the variational problem represented by Eqs. (II-1) and (II-2), one additional random variable may need to be considered, namely the initial state x_0. The sum total of the uncertainties encountered so far is hence characterized by the variables x_0, w, and p.

If the variables x_0, w, and p were perfectly known, the stated optimization problem could be solved by well-known deterministic techniques and a control $u(t)$ minimizing the cost J could be found. If the same control $u(t)$ were applied in the presence of uncertainty (*i.e.*, if x_0, w, and p are random variables) the resulting cost would also be a random variable. In this situation, it is customary to minimize the expected cost $E\{J\}$ by finding the appropriate control u. This implies that a stochastic optimization problem must be solved, which in general is very difficult. The approach we take in Part C below deliberately avoids the general stochastic optimization problem. Broadly speaking, we first ask the question "What is the degrading effect upon performance of these uncertainties?" If the degrading effect is acceptably small, the design is satisfactory; if not, compensation of the control signal is introduced to reduce the degrading effects of uncertainty. To derive these compensatory signals, the well-known and easy to implement results of linear control theory are liberally used.

C. Deterministic Phase

1. Discussion

Having tentatively decided on a reasonable model, the second step of the methodology consists of optimizing this model without considering any uncertainties x_0, p, or w. The resulting deterministic optimization problem is stated as follows:

$$\min_{u(t) \in U} \int_0^T l(x,u,t)dt + \Phi(x_T, T) \quad , \tag{II-7a}$$

subject to the differential equation constraints

$$\dot{x} = f(x,u,\bar{p},\bar{w},t) \quad ; \quad x\big|_{t=0} = \bar{x}_0 \quad , \tag{II-8a}$$

where the random variables x_0, p, and w have been replaced by their means $\bar{x}_0$, $\bar{p}$, and $\bar{w}$. The minimum cost obtained as a result of this deterministic optimization is denoted by J°; this cost is clearly a function of $\bar{x}_0$, $\bar{p}$, and $\bar{w}$.

Depending on the nature of the optimization procedures chosen to carry out this step, the optimum deterministic control u° is obtained either as a function of time $u^\circ(t)$—a control schedule—or as a function of state and time $u^\circ(x, t)$—a control law. Gradient procedures naturally lead to control schedules, whereas dynamic programming naturally leads to control laws.

If there were no uncertainties, *i.e.*, if the variables x_0, p, and w were exactly equal to the means $\bar{x}_0$, $\bar{p}$, and $\bar{w}$ assumed for the optimization, the closed- and open-loop configurations would be completely equivalent. Since in actuality these uncertainties exist, the performance obtainable with both configurations may be quite different. The deviation of performance from J° as a result of these uncertainties is most conveniently assessed by means of *sensitivity theory*[4] for those problems where the required linearizations are valid. With the help of sensitivity theory, closed-loop performance can be calculated readily, even if the result of the deterministic optimization is a control schedule. In view of this, we assume that an open-loop solution $u^\circ(t)$ of the stated deterministic optimization problem has been obtained.

For the static case, the optimization problem is stated as

$$\min_u F(x, u) \quad , \qquad \text{(II-7b)}$$

subject to the algebraic constraints

$$g(x, u, p, w) = 0 \quad , \qquad \text{(II-8b)}$$

where again the uncertainties p and w are replaced by their means $\bar{p}$ and $\bar{w}$. The resulting minimum deterministic cost J° is a function of $\bar{p}$ and $\bar{w}$; the optimum control u° is a number which also depends on $\bar{p}$ and $\bar{w}$.

To complete the deterministic phase, it is necessary to establish the sensitivity properties of J° with respect to the uncertainties x_0, p and w. This can be accomplished either by simulation or by analytical procedures, notably those developed in Ref. 4 and repeated in Sec. III of this report.

The most frequently used analytical approach to determine the sensitivity properties is to perform a Taylor series expansion of the cost J° with respect to the variables x, u, x_0, p, and w about the *nominal solution* defined by $\bar{x}_0$, u°, $\bar{p}$, and $\bar{w}$. The constraining equations (II-2) and (II-4) prescribe the variation Δx of the dependent (state) variable

x resulting from any variation Δu, Δx_0, Δp, and Δw of the remaining problem variables. To summarize, the variation in cost ΔJ—a positive or negative scalar—can be expressed by means of a Taylor series expansion in terms of the variations Δx and Δu, which in turn are constrained by the equations of motion (II-2) or (II-4).

The importance of carrying out the Taylor series expansion to sufficiently high order has been pointed out in Ref. 4. For the purposes of this discussion, it suffices to make the following comments:

(1) If only the first-order terms of the Taylor series expansion are retained, *i.e.*, when ΔJ is expressed linearly in terms of Δx and Δu, then the degrading or beneficial effects of actual variations Δx_0, Δp and Δw upon J° are obtained.

(2) By retaining the second-order expansion terms, a quadratic model for ΔJ supplemented by a linear model of the constraining equations results. This model can be used to determine the degrading effects upon performance of the changes Δx_0, Δp, and Δw when not compensated by a suitable change Δu in the control. This same model can be used to determine the optimum change Δu to accommodate observed changes as well as uncertainties in the variables, Δx, Δp, and Δw from the linear theory of optimum control and estimation. This second-order sensitivity model is probably the most important one to be considered for the design methodology under discussion, since it strikes a reasonable compromise between complexity and accuracy.

(3) By modification of the nominal control, the stochastic and dual effects discussed can be accommodated, as will be shown in Sec. III.

2. Example

To clarify these ideas, the following static optimization problem discussed in Ref. 8 is treated:

It is desired to select the speed u of a supersonic aircraft in level flight such that the fuel expended per mile of travel is minimized. The cost function is

$$J = \frac{\sigma T}{cM} \tag{II-9}$$

where

σ = specific fuel consumption = $0.29 \cdot 10^{-3}$ lb s^{-1}/lb of thrust

T = engine thrust, in lb

c = speed of sound = 968.1 ft/s at the prescribed altitude of 50,000 ft

M = Mach number, here identified with the control variable u.

The constraining equations are

$$
\begin{aligned}
L - mg + T \sin(\alpha + \epsilon) &= 0 \\
D - T \cos(\alpha + \epsilon) &= 0 \quad , \qquad \text{(II-10)}
\end{aligned}
$$

where

L = lift = $C_{L_\alpha} \alpha \dfrac{\rho c^2 M^2}{2} S$

D = drag = $(C_{D_0} + \eta C_{L_\alpha} \alpha^2) \dfrac{\rho c^2 M^2}{2} S$

α = angle of attack, in radians

ϵ = fixed angle determined by the aircraft geometry = 0.05 rad

ρ = air density = $361.8 \cdot 10^{-6}$ slugs/ft^3 at the prescribed altitude of 50,000 ft

S = wing area = 530 ft^2

mg = weight = 34,000 lb

C_{L_α}, C_{D_0}, η = aerodynamic parameters, which vary with mach number M as shown in Fig. II-1.

Notation: The variables x, u, and p of Eqs. (II-3) and (II-4) are identified with the following variables in the example

$$
\begin{array}{llll}
u = M & x_1 = T & p_1 = mg \\
 & x_2 = \alpha & p_2 = S
\end{array}
$$

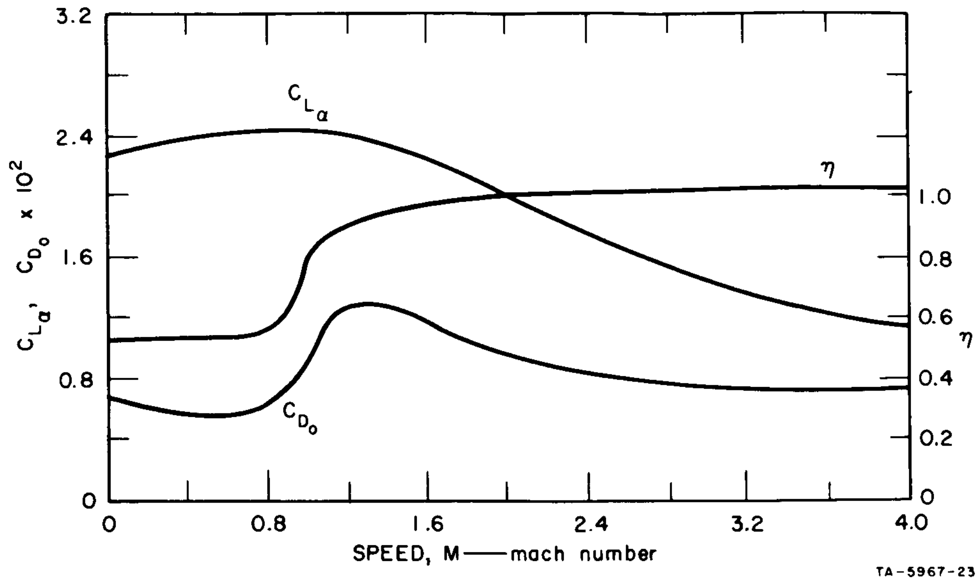

FIG. II-1 VARIATION OF C_{L_α}, C_{D_o}, AND η WITH M

3. Deterministic Optimization

For the nominal values of mg = 34,000 lb and S = 530 ft^2, the variation of J with mach number M is shown in Fig. II-2. The optimum mach number is 2.863 for which

$$T = 6.133 \text{ lb}$$

$$\alpha = 5.716 \cdot 10^{-4} \text{ rad}$$

$$J = 6.847 \cdot 10^{-4} \text{ lb/ft} \cong 3.423 \text{ lb/mile}$$

4. First-Order Perturbation Model

It is assumed that the variables u, x, and p are allowed to vary by small amounts Δx, Δu, and Δp and it is desired to find the resulting first-order variation ΔJ of cost; the variations Δx, Δu, and Δp cannot be chosen arbitrarily, but must continue to satisfy the constraining Eq. (II-10), which effectively means that the dependent variation Δx can be expressed in terms of the independent variations Δu and Δp and eliminated from the expression for ΔJ, which thus becomes*

$$\Delta J = A_1 \Delta u + A_2 \Delta p \quad . \tag{II-11}$$

* The sensitivity equations used in this section are substantiated in Appendix A.

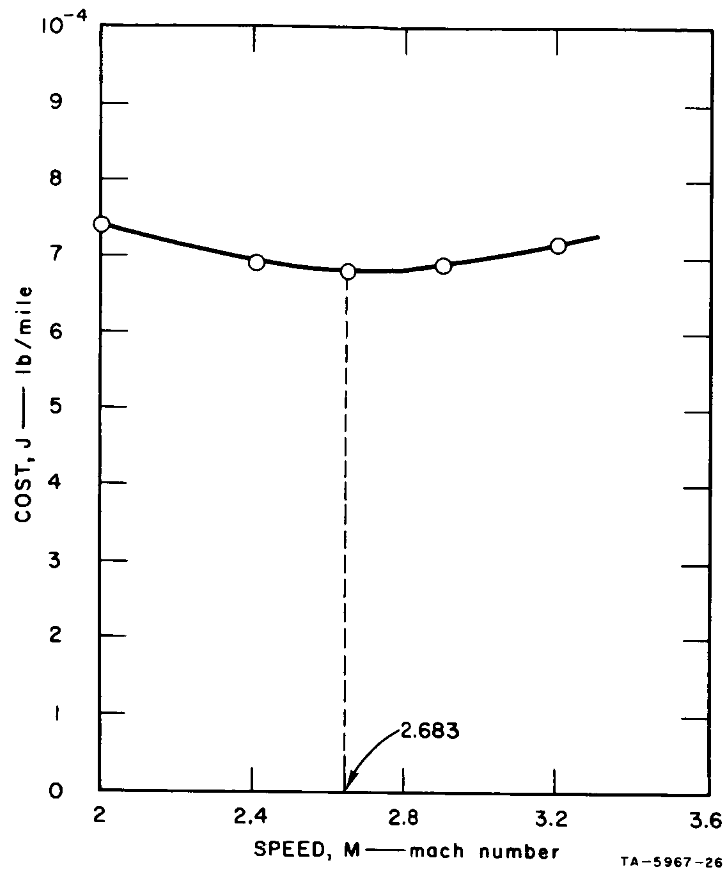

FIG. II-2 VARIATION OF J WITH SPEED AT CONSTANT ALTITUDE OF 50,000 ft

For a weight variation $\Delta p_1 = \Delta mg$, the corresponding sensitivity coefficient A_2 is $+7.298 \cdot 10^{-9}$. As expected, A_1 is zero, since u was chosen to be optimal.

5. Second-Order Perturbation Model

It is again assumed that the variables u, x, and p are allowed to vary by small amount Δu, Δx and Δp subject to the constraining Eq. (II-8). It is now desired to express the resulting cost variation ΔJ in terms of the independent variations Δu and Δp by considering terms up to second order. The variation ΔJ now becomes

$$\Delta J = A_1 \Delta u + A_2 \Delta p + \Delta u' B_{11} \Delta u + \Delta u' B_{12} \Delta p + \Delta p' B_{22} \Delta p \qquad \text{(II-12)}$$

where A_1 and A_2 remain unchanged and where the matrices B_{11}, B_{12} and B_{22} are

$$B_{11} = 1.100 \cdot 10^{-4}$$

$$B_{12} = [-5.745 \cdot 10^{-9} \quad 3.686 \cdot 10^{-7}]$$

$$B_{22} = \begin{bmatrix} 1.076 \cdot 10^{-13} & -0.690 \cdot 10^{-11} \\ -0.690 \cdot 10^{-11} & 4.427 \cdot 10^{-10} \end{bmatrix} \qquad \text{(II-13)}$$

In the static optimization example under discussion, we treat the parameters p_1 = mg and $p_2 = S$ in an identical fashion, although in an automatic cruise control system, weight acts as a state variable and wing surface is an inaccurately known parameter. By analogy with the dynamic optimization problem, we therefore refer to that part of the controller which compensates for weight deviations Δp_1 as "closed-loop" and to that part of the controller which compensates for identified deviations in p_2 as "adaptive." It is noteworthy that the approach taken allows one to treat the closed-loop and the adaptive problem in exactly the same manner. The adaptive system is simply viewed as a closed-loop system in which additional inaccurately known quantities are monitored and compensated for by the controller.

D. Stochastic Analysis Phase

1. Discussion

The stochastic phase consists of modeling the uncertainties by appropriate probability density functions and of assessing the degrading effects upon performance of these uncertainties for various possible control system implementations.

Encoding of the uncertainties that affect the variables x_0, p, and w by probability density functions can be done on the basis of actual measurement by consideration of known physical laws or by well-planned interviewing procedures. For example: Wind-induced perturbations can be measured, electronic circuit noise can be related analytically to temperature, and the tolerance on a plant parameter p can be obtained from the plant designer. For analytical convenience, these probability density functions are often taken to be Gaussian.

The simplest controller implementation to be analyzed from the point of view of sensitivity is the open-loop configuration. For this case, $\Delta u \equiv 0$ and the change in performance ΔJ is obtained from a Taylor series expansion with $\Delta u \equiv 0$. Since the deviations Δx_0, Δp, and Δw are random variables, ΔJ is also a random variable. The mean $E\{\Delta J\}$, which can be easily computed, is a good measure of the performance change caused by the uncertainties Δx_0, Δp, and Δw.

The next most common controller implementation is the closed-loop optimum configuration. Here, the control u° is made a function of the state and time, *i.e.*,

$$u^\circ = g(x,t) \qquad \Delta u = G\Delta x \tag{II-14}$$

Substitution of Eq. (II-14) into Eq. (II-10) determines the variation, Δx caused by the perturbations Δx_0, Δp and Δw. The mean $E\{\Delta J]$, which again can be calculated readily,is in general different for the open- and closed-loop configurations.

As a next and very realistic step in complexity, we may assume that the state x is not measured accurately or completely. The practical implementation of the system, shown in Fig. II-3 now contains an instrumentation subsystem followed by an estimator, which may or may not be optimal. These two constituents can be described by a "law of estimation" of the form

$$\dot{\hat{x}} = \hat{f}(\hat{x}, x, v, t) \quad , \tag{II-15}$$

where $\hat{x}$ is the estimator output and where the random variable v denotes the measurement noise. Customarily, the law of control (II-14) is now replaced by

$$u^\circ = g(\hat{x}, t) \quad ; \tag{II-16}$$

that is, the estimate $\hat{x}$ is used in lieu of the true state x. Linearization of the laws of estimation and control provide the expressions

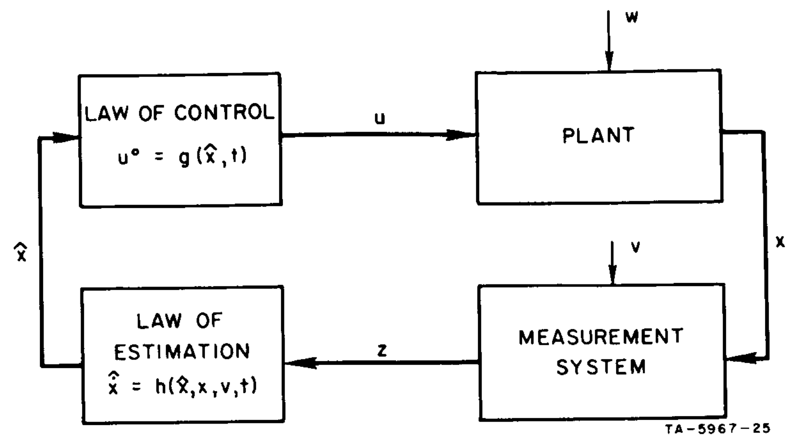

FIG. II-3 CLOSED-LOOP CONTROL SYSTEM WITH MEASUREMENT NOISE AND ESTIMATION

$$\Delta\dot{\hat{x}} = H\Delta\hat{x} + M\Delta x + v \tag{II-17}$$

$$\Delta u = G\Delta\hat{x} \tag{II-18}$$

and substitution of (II-18) into the linearized equations of motion yields the variations Δx and Δu in terms of the deviations Δx_0, Δp, Δw, and v. Substitution of Δx and Δu into the cost function yields ΔJ, the mean $E\{\Delta J\}$ of which can be computed readily.

The expressions $E\{\Delta J\}$ corresponding to the open-loop and closed-loop configurations with and without measurement noise tell the designer in a quantitative fashion to what extent the performance of a deterministic optimum design will degrade as a result of uncertainty for three common design configurations, namely:

(1) The open-loop optimum deterministic configuration

(2) The closed-loop optimum deterministic configuration

(3) The closed-loop configuration with optimum deterministic law of control in which the actual state has been replaced by the estimate $\hat{x}$, which may or may not be optimal. This configuration is shown in Fig. II-3.

If $E\{\Delta J\}$ is sufficiently small for one such configuration, a satisfactory design has been achieved. If $E\{\Delta J\}$ is unacceptably large, the design methodology proceeds to the stochastic optimization phase.

2. Example

Proceeding with the aircraft cruise control example previously used, we analyze the expected degradation of performance corresponding to the two following situations:

(1) The cruise-control system is an open-loop configuration, for which the scheduled weight variation mg(t) is precomputed. The standard deviation of this precomputed weight information is assumed to be 1000 lb.

(2) The cruise-control system is a closed-loop configuration in which the actual weight is sensed at all times with perfect accuracy and the control u is adjusted accordingly.

For both cases, it is assumed that the wing area S is known with perfect accuracy.

From the results previously given [see Eqs. (II-11) and (II-12)] it follows that

$$\Delta J = 7.298 \cdot 10^{-9}\Delta p_1 + 1.100 \cdot 10^{-4}\Delta u^2 - 5.745 \cdot 10^{-9}\Delta u\Delta p_1 + 1.076 \cdot 10^{-13}\Delta p_1^2 \quad . \tag{II-19}$$

For the open-loop cruise control system, $\Delta u = 0$ and the expected variation of cost is

$$E\{\Delta J\}^{OL} = E\{7.298 \cdot 10^{-9}\Delta p_1 + 1.076 \cdot 10^{-13}\Delta p_1^2\} = 1.076 \cdot 10^{-7} \text{ lb/ft} \quad .$$

For the closed-loop configuration with perfect weight information, the optimum law of control

$$\Delta u = G\Delta p_1$$

is chosen so as to minimize the cost variation ΔJ, that is

$$\Delta u = 2.612\ 10^{-5}\Delta p_1 \quad . \tag{II-20}$$

Thereafter, the cost variation ΔJ is given by

$$\Delta J = 7.298\ 10^{-9}\Delta p + 0.328 \cdot 10^{-13}\Delta p_1^2$$

and the expected cost variation $E\{\Delta J\}$ becomes

$$E\{\Delta J\}^{CL} = 0.328 \cdot 10^{-7} \text{ lb/ft} \quad .$$

The two values obtained for $E\{\Delta J\}$ may now be interpreted as follows:

In the closed-loop case with perfect information ($E\{\Delta J\}^{CL} = 0.328 \cdot 10^{-7}$) the control Δu has been optimally adjusted to the variations Δp_1 and the expected variation of cost cannot be reduced any further as a result of control.* $E\{\Delta J\}^{CL}$ thus establishes a lower bound of cost, when there exists an uncertainty of the type Δp_1, which effectively plays the role of plant noise.

In the open-loop case

$$E\{\Delta J\}^{OL} = 1.076\ 10^{-7} > E\{\Delta J\}^{CL} = 0.328 \cdot 10^{-7} \quad .$$

The unnecessary increase of cost W resulting from the absence of any correction Δu is thus

$$W = E\{\Delta J\}^{OL} - E\{\Delta J\}^{CL} = 0.748 \cdot 10^{-7} \quad .$$

If the magnitude of W is tolerable, an open-loop cruise control system is entirely adequate; if not, a reduction of W by means of the more refined control system configurations to be discussed in Part E below must be attempted.

It is repeated at this point that the quantity $E\{\Delta J\}^{CL}$ represents a lower bound; if its magnitude is not tolerable, the only remedy is to reduce the plant noise Δp_1 or to redesign the plant.

E. Stochastic Optimization Phase

1. Discussion

In the course of the stochastic optimization phase, the designer attempts to supplement the control schedule $u^{\circ}(t)$ obtained in the course of the deterministic optimization so as to reduce the degrading effects

* This statement applies to the second-order perturbation model under discussion in this section. Additional improvements may conceivably be obtained by use of higher-order models.

of uncertainty by improved data processing. To accomplish this, the following approaches, listed in increasing order of complexity, are available.

(1) Supplement the control schedule $u^{\circ}(t)$ by an optimum law control relating Δu to Δx.

(2) Supplement the control schedule $u^{\circ}(t)$ by optimum laws of estimation and control; that is, determine the optimum estimate $\Delta\hat{x}$ of Δx and relate Δu to $\Delta\hat{x}$.

(3) Supplement the control schedule $u^{\circ}(t)$ by a correction Δu to compensate for identified parameter deviations Δp; in other words, design an *adaptive* system.

(4) Analyze the stochastic and dual effects and supplement the original schedule $u^{\circ}(t)$ by a correction schedule $\Delta u(t)$ in addition to the remedies discussed in (1) to (3) above.

2. Example

For the cruise control problem discussed previously, we have already calculated the expected cost variation $E\{\Delta J\}^{CL}$ for closed-loop control with perfect information on Δp_1, and we have established the law of control as

$$\Delta u = 2.612 \cdot 10^{-5} \Delta p_1 \quad .$$

We may now depart from the idealized situation of noiseless measurements and assume that the actual variation Δp_1 is measured by a sensor the standard deviation of which is 200 lb; that is, the reading of the sensor is $\Delta p_1 + v$, where v is the measurement noise. In actual practice, the true weight of the aircraft is not known perfectly, because the fuel flow gauges accumulating the weight of the fuel burnt are not completely accurate.

For the closed-loop system with imperfect weight information, the waste W is related to the measurement v of standard deviation $\sigma_v = 200$ lb in Appendix A as

$$W = \Omega_{11}\sigma_v^2 = 0.3 \cdot 10^{-8} \quad .$$

If this added cost W is excessive, we may either use a better instrument (the standard deviation of v is less than 200) or else we may estimate the true weight more accurately. The simplest approach would consist of time-averaging the accumulated readings of the fuel gauges, whereas a more effective approach would consist of utilizing in addition to the readings of the fuel gauges the information contained in the constraining Eq. (II-13). For example, if the thrust T and the speed M are measured, the weight mg can be inferred from (II-13).

As a result of estimation, the measurement noise v is effectively reduced and the resulting W can be calculated exactly as was done before.

To illustrate the *adaptive correction*, we may assume that some aerodynamic parameter, for example p_2 = , is not accurately known. From the second-order perturbation model of Eq. (II-19), we can derive an optimum law of adaptation in exactly the same manner that the optimum law of control was previously derived. The result is

$$\Delta u = -1.675 \cdot 10^{-3} \Delta p_2 \quad .$$

The improvement in cost it allows is established in exactly the same manner that closed-loop control was previously justified.

Generally speaking it should be noted that, with the perturbation model considered here, adaptation is treated exactly like closed-loop control: In the closed-loop configuration, the correction Δu is made to depend on an observed variation of the plant state, whereas in the adaptive configuration, an additional correction Δu is made to depend on an observed variation of the plant model.

For the simple example chosen to illustrate this section, the stochastic and dual effects might enter as follows:

Stochastic Effect: Let us assume that instead of the quadratic perturbation model of Eq. (II-19), we use a more elaborate model of the general form

$$\Delta J = f(\Delta u, \Delta p_1) \quad . \qquad \text{(II-21)}$$

We assume for illustrative purposes that the random variable Δp_1 of mean $m = 0$ and standard deviation σ is not measured. The problem consists of finding a correction Δu such that the expected cost variation $E\{\Delta J\}$ is minimized. If the function f in Eq. (II-21) is quadratic, then Δu is zero, since u° was an optimal deterministic control. If f has a different form, then there will in general be a correction Δu which reflects this stochastic effect.

Dual Effect: This effect consists of deliberately departing from the optimum deterministic control u° in order to acquire more information about Δp. A possible manifestation of the dual effect in this example might consist of perturbing the optimum speed u° by Δu in order to estimate the weight mg and/or the wing surfaces more accurately. To be more specific, we may assume that the variables M and T are measured and we want to estimate the parameters mg and S from the model (II-10). In a first experiment, we select $M = M_1$; a thrust $T = T_1$ and an angle of attack $\alpha = \alpha_1$ follow. Equations (II-10) are written compactly in terms of the remaining unknowns as

$$g_1(\alpha_1, \mathrm{mg}, S) = 0$$

$$g_2(\alpha_1, \mathrm{mg}, S) = 0 \quad . \qquad \text{(II-22)}$$

Since there are three unknowns and only two equations (II-22), an additional experiment must be performed with $M = M_2$, $T = T_2$ and $\alpha = \alpha_2$; the following two additional equations are now obtained

$$g_1(\alpha_2, \mathrm{mg}, S) = 0$$

$$g_2(\alpha_2, \mathrm{mg}, S) = 0 \quad . \qquad \text{(II-23)}$$

The four equations (II-22), (II-23) suffice to solve for the four unknowns α_1, α_2, mg, and S. To obtain the set (II-23), M had to depart from its optimum value, which caused a temporary expenditure of fuel.

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III AN APPROXIMATE METHOD FOR COMPUTING THE PERFORMANCE OF A STOCHASTIC, NONLINEAR CONTROL SYSTEM WITH APPLICATIONS TO SENSITIVITY AND OPTIMIZATION THEORY

A. Introduction

In this section, the equations needed to apply to dynamic systems the systematic procedure for designing control systems presented in the preceding section are derived.

The first step in the procedure is determination of the optimal deterministic control, either in the form of an open-loop control schedule or a closed-loop control law. By linearizing the necessary conditions about the nominal trajectory defined by the control schedule that is optimal for a nominal initial condition, Breakwell, Speyer, and Bryson[9] obtained a linear approximation to the optimal closed-loop control law, which is valid in a neighborhood of the nominal trajectory. In Sec. III-A an approximate expression for the performance of a control system, in which the controller consists of a control law that is a nominal control schedule plus a time-varying linear function of the difference between the estimated and nominal plant state and an estimator that is characterized by a second time-varying linear gain, is derived by application of the Hamilton-Jacobi equation. The novelty of this development is its consideration of stochastic effects and suboptimal control and estimation.

Computation of the sensitivity of system performance to disturbance inputs, measurement noise, and parameter uncertainty is the second step of the procedure outlined in Sec. II. In the predecessor contract[5] an expression was obtained for the performance of discrete-time linear systems with Gaussian disturbances and quadratic cost that clearly displays the sensitivity to disturbances and measurement noise. In Ref. 10, analogous results are presented for the continuous-time case. These results are extended to nonlinear systems and parameter uncertainties in Sec. III-B by use of the results of Sec. III-A.

As Kushner[11] discovered, the deterministic optimum control schedule is not necessarily the best nominal control schedule for use in the controller described above; furthermore, the feedback and estimator gains may also need to be varied to obtain optimum performance when parameter variations are present but not estimated. A deterministic optimal control problem whose performance index is the expression for performance derived in Sec. III-A is formulated in Sec. III-C in such a manner that the solution provides the values of the nominal control, feedback gain, and estimation gain that are optimal in the regions for which the approximations involved are valid. It is shown how such a problem may be solved by the gradient method; similar problems in which sensitivity costs are added to the performance index have been considered by Tuel[12] and D'Angelo, Moe, and Hendricks,[13] but their sensitivity measurement is not derived from the original performance index.

B. Computation of Performance

In this part, perturbation theory is applied to the Hamilton-Jacobi equation to derive an expression accurate to terms of second degree for the performance of a nonlinear, stochastic control system.

1. System Description

a. Plant

The plant to be controlled is described by the state equation

$$\dot{x} = f(x, u, t) + w \quad , \tag{III-1}$$

where* x is the plant state, u is the control, and w is the disturbance

$$E(w) = 0, \quad E[w(t)w^T(\tau)] = \hat{Q}(t)\delta(t - \tau) \quad ,$$

$$E[x(0)] = \hat{x}(0), \quad E\{[x(0) - \hat{x}(0)][x(0) - \hat{x}(0)]^T\} = \hat{P}(0) \quad ,$$

and by the measurement equation

$$z = Hx + v \quad , \tag{III-2}$$

* In this presentation arguments of functions will be suppressed unless it is necessary to display them.

where z is the measurement, v the measurement noise,

$$E(v) = 0, \quad E[v(t)v^T(\tau)] = \hat{R}(t)\delta(t - \tau) \quad .$$

The performance index is

$$J = E\{\int_0^T l(x,u,t)dt + \Phi[x(T),T]\} \tag{III-3}$$

b. Controller

The controller consists of a control law

$$u = u^\circ - K(\hat{x} - x^\circ) \quad , \tag{III-4}$$

where the nominal control u° and nominal state x° obey

$$\dot{x}^\circ = f(x^\circ, u^\circ, t) \quad ; \tag{III-5}$$

and an estimator described by*

$$\dot{\hat{x}} = f(\hat{x},u,t) + \hat{K}(t)(z - H\hat{x}) + 1/2 f^\circ_{xx}\, o\hat{P} \quad , \tag{III-6}$$

where $\hat{x}$ is the estimate of the plant state and $\hat{P}$ is the conditional covariance of the estimate*

$$\hat{P} \triangleq E\{(\hat{x} - x)(\hat{x} - x)^T/Z\} \quad , \tag{III-7}$$

$$Z(t) \triangleq \{z(\tau) : 0 \leq \tau \leq t\} \quad ,$$

$$(f^\circ_{xx}\, o\hat{P})^{(i)} \triangleq \sum_{j,k} \left.\frac{\partial^2 f^{(i)}}{\partial x^{(j)}\partial x^{(k)}}\right|_{x^\circ, u^\circ} \hat{P}^{(j,k)} \quad .$$

An equation for $\hat{P}$ will be given below.

* Superscripts in parentheses refer to components; "o" is used to denote quantities evaluated along the nominal trajectory. The necessity for the last term in Eq. (III-6) is apparent from Eq. (B-2) in Appendix B.

2. Estimation

The conditional probability density of the state of the plant, which Wonham,[14] Meier,[5] Mortensen,[15] and others have pointed out is the optimal estimator for control purposes, can be computed using Bayes rule in the discrete time case[16] and by use of a generalized Fokker-Planck equation in the continuous time case.[15,17] An approximate equation for the conditional mean that takes the form of Eq. (III-6) with

$$\hat{K} = \hat{P}H^T\hat{R}^{-1} \tag{III-8}$$

can be obtained by application of perturbation theory to the Fokker-Planck equation.[17]

In Appendix B-1 it is shown that

$$E[x(t) - \hat{x}(t)/Z(t)] \approx 0 \quad ; \tag{III-9}$$

hence $\hat{x}$ is an unbiased estimator. An approximate equation for $\hat{P}$ is also derived in this appendix:

$$\dot{\hat{P}} \approx (f_x^o - \hat{K}H)\hat{P} + \hat{P}(f_x^o - \hat{K}H) + \hat{Q} + \hat{K}\hat{R}\hat{K} \quad . \tag{III-10}$$

Because f_x^o is evaluated along the nominal trajectory, $\hat{P}$ may be computed *a priori*; $\hat{P}$ will be smallest for the choice of $\hat{K}$ given by (III-8).

3. Control

Using a limiting argument on the dynamic programming functional equation for discrete time stochastic control problems, Kushner[18] derived a generalized Hamilton-Jacobi equation for solving stochastic control problems that can be applied to the problem described in Eqs. (III-1) to (III-7) as follows: The whole system, plant, and controller, is taken as a fictitious plant to be controlled whose state may be measured exactly, but over which no control may be exerted.

The use of this technique to approximately compute the performance of the stated problem is given in Appendix B-2; the results are

$$J = J_d + \lambda^T(\hat{x} - x^o) + [\hat{x}(0) - x^o(0)]^T P(0)[\hat{x}(0) - x^o(0)]$$

$$+ \operatorname{tr}[P(0)\hat{P}(0)] + \int_0^T \operatorname{tr}[P\hat{Q} + P^*\hat{P} + 2\overline{P}(\hat{K}\hat{R} - \hat{P}H^T)\hat{K}^T]\,dt \quad ,$$

(III-11)

where J_d is the nominal deterministic cost given by

$$J_d = \int_0^T l(x^o, u^o, t)\,dt + \Phi[x^o(T), T] \quad ; \qquad \text{(III-12)}$$

the adjoint variable λ obeys

$$-\dot{\lambda} = (H_x^o - H_u^o K)^T, \quad \lambda(T) = \Phi_x^o \quad ; \qquad \text{(III-13)}$$

where H is the Hamiltonian function

$$H = l + \lambda^T f \quad .$$

The cost matrices P and P^* obey

$$-\dot{P} = Pf_x^o + f_x^{oT} P + \frac{1}{2} H_{xx}^o - P^*, \quad P(T) = \frac{1}{2}\Phi_{xx}^o \quad ,$$

$$P^* = \left(Pf_u^o + \frac{1}{2}H_{xu}^o\right)K + K^T\left(Pf_u^o + \frac{1}{2}H_{xu}^o\right)^T - \frac{1}{2}K^T H_{uu} K \quad ;$$

(III-14)

and the cost matrix $\overline{P}$ obeys

$$-\dot{\overline{P}} = \overline{P}(f_x^o - \hat{K}H) + (f_x^o - f_u^o K)^T \overline{P} - \left(Pf_u^o + \frac{1}{2}H_{xu}^o\right) K + \frac{1}{2}K^T H_{uu}^o K \quad ,$$

$$\overline{P}(T) = 0 \quad . \qquad \text{(III-15)}$$

Some comments about these results are in order:

(1) If the nominal control u^o is the deterministic optimum, then $H_u^o = 0$ and Eq. (III-13) reduces to the familiar form of the adjoint equation.

(2) For the optimal estimation gain given by Eq. (III-8) the third term in the integrand of Eq. (III-11) is zero.

(3) If the optimal control gain

$$K = 2(H^{\circ}_{uu})^{-1}\left(Pf^{\circ}_{u} + \frac{1}{2}H^{\circ}_{xu}\right)^{T}$$

is used then $\overline{P} \equiv 0$ and this third term again drops out.

(4) For the linear quadratic case these results are exact and were previously derived in a different manner by Meier and Anderson.[10]

C. Sensitivity

This section presents the application of the theory developed in Part B, to computation of sensitivities.

1. System Description

It is desired to determine the sensitivity to disturbances, measurement noise and parameter variations of a control system described by Eqs. (III-1) to (III-7) with $f(x,u,t)$ replaced by $f(x,u,\alpha,t)$ and $l(x,u,t)$ replaced by $l(x,u,\alpha,t)$, where α is the parameter vector. Furthermore, u° is taken as the deterministic optimum and $\hat{K}$ and K are given the optimum values, within the validity of the approximations, of

$$\hat{K} = \hat{P}\hat{H}\hat{R}^{-1}$$

$$K = 2H^{\circ -1}_{uu}\left(f^{\circ T}_{u}P + \frac{1}{2}H^{\circ}_{ux}\right) , \qquad \text{(III-16)}$$

where $\hat{P}$ is given by Eq. (III-10) and P is given by Eq. (III-14).

2. Augmentation of the State Vector and Partitioning of the Matrices

In order to compute sensitivities with respect to α, the state vector must be augmented:

$$\mathbf{X} = \begin{bmatrix} x \\ \cdots \\ \alpha \end{bmatrix} \qquad \text{(III-17)}$$

Since α is constant, but unknown, $\dot{\alpha} = 0$; the differential equation for the augmented state is

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, u, t) + w = \begin{bmatrix} f(x, u, \alpha, t) \\ \hline 0 \end{bmatrix} + \begin{bmatrix} w \\ \hline 0 \end{bmatrix} , \quad \mathbf{X}(0) = \begin{bmatrix} x(0) \\ \hline \alpha(0) \end{bmatrix} . \tag{III-18}$$

Similarly, the measurement equation, control law, and estimation equation may be written for the augmented state

$$z = \mathbf{H}\mathbf{X} + v$$

$$u = u^{\circ} - \mathbf{K}(\hat{\mathbf{X}} - \mathbf{X}^{\circ})$$

$$\hat{\mathbf{X}} = \mathbf{f}(\hat{\mathbf{X}}, u, t) + \hat{\mathbf{K}}(z - \mathbf{H}\hat{\mathbf{X}}) + \frac{1}{2}\mathbf{f}^{\circ}_{\mathbf{XX}}\hat{\mathbf{P}} , \tag{III-19}$$

where

$$\hat{\mathbf{X}} = \begin{bmatrix} \hat{x} \\ \hline \hat{\alpha} \end{bmatrix} , \quad \mathbf{X}^{\circ} = \begin{bmatrix} x^{\circ} \\ \hline \alpha^{\circ} \end{bmatrix} ,$$

$$\mathbf{H} = [H|0] , \quad \mathbf{K} = [K|K_{\alpha}] , \quad \hat{\mathbf{K}} = \begin{bmatrix} \hat{K} \\ \hline 0 \end{bmatrix} ,$$

and

$$\dot{\mathbf{X}}^{\circ} = \mathbf{f}(\mathbf{X}^{\circ}, u^{\circ}, t) , \quad \mathbf{X}^{\circ}(0) = \hat{\mathbf{X}}(0) .$$

Because of the form of $\hat{\mathbf{K}}$, the estimate $\hat{\alpha}$ of the parameter vector will equal the nominal value α° (*i.e.*, no estimation takes place); hence, the performance of the system will be independent of the choice of K_{α}. In Sec. III-D, adaptive situations, in which α is actually estimated, will be considered. Open-loop control may be considered as mathematically equivalent to infinite measurement noise.

Equations (III-18) and (III-19) describe a system exactly like that given in Sec. III-B in Eqs. (III-1) to (III-7), eccept that bold-face type replaces ordinary type. In the following paragraphs, the equation for performance given in Sec. III-B is applied to the bold-face system and the results interpreted in terms of sensitivity theory.

3. Solution of the Control and Estimation Equations

The matrix $\mathbf{P}$ corresponding to the augmented system can be partitioned as follows

$$\mathbf{P} = \left[\begin{array}{c|c} P_x & P_{x\alpha} \\ \hline P^T_{x\alpha} & P_\alpha \end{array}\right] . \tag{III-20}$$

The symbol P_x does not indicate partial differentiation with respect to x, etc. In Appendix (B-3) equations are given for the parts of $\mathbf{P}$ by substitution of Eq. (III-20) into Eq. (III-14) printed in bold-face type. Because of the form of $\mathbf{f}$, the equation obtained for P_x is identical with Eq. (III-14) printed in ordinary type.

If $\mathbf{K}$ is taken as

$$\begin{aligned} \mathbf{K} &= 2\mathbf{H}^{o\,-1}_{uu}\left(\mathbf{f}^{o\,T}_{u}\mathbf{P} + \frac{1}{2}\mathbf{H}^{o}_{ux}\right) \\ &= \left[2H^{o\,-1}_{uu}\left(f^{o\,T}_{u}P_x + \frac{1}{2}H^{o}_{ux}\right) \Big| \, 2H^{o\,-1}_{uu}\left(f^{o\,T}_{u}P_{x\alpha} + \frac{1}{2}H^{o}_{u\alpha}\right)\right] \\ &= [K|K_\alpha] \quad , \end{aligned} \tag{III-21}$$

$\overline{\mathbf{P}}$ will be identically zero. This is possible, since the choice of K_α is arbitrary as explained above.

Similarly, $\hat{\mathbf{P}}$ may be partitioned into

$$\hat{\mathbf{P}} = \left[\begin{array}{c|c} \hat{P}_x & \hat{P}_{x\alpha} \\ \hline \hat{P}^T_{x\alpha} & \hat{P}_\alpha \end{array}\right] . \tag{III-22}$$

Since Eq. (III-10) for $\hat{\mathbf{P}}$ is linear, the superposition principle may be applied:

$$\hat{\mathbf{P}}(t) = \left[\begin{array}{c|c} \hat{P}_x(t) & 0 \\ \hline 0 & 0 \end{array}\right] + \theta(t) \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \hat{P}_\alpha(0) \end{array}\right] \theta^T(t) \tag{III-23}$$

where $\hat{P}_x$ is the solution of Eq. (III-10) printed in ordinary type and*

$$\dot{\theta} = (f_x^\circ - \hat{K}H)\theta \quad , \quad \theta(0) = I \quad . \tag{III-24}$$

Substitution of Eq. (III-23) into Eq. (III-10), this time printed in bold-face type, will show that the form of $\hat{\mathbf{P}}$ is a solution. θ may be partitioned into

$$\theta = \left[\begin{array}{c|c} \theta_x & \theta_{x\alpha} \\ \hline \theta_{\alpha x} & \theta_\alpha \end{array}\right] \quad . \tag{III-25}$$

Equations for the parts of θ are given in Part 3 of Appendix B.

4. <u>The Sensitivity Relations</u>

When Eq. (III-11) is written in terms of bold-face quantities and the partitioned forms substituted, the following expression for performance results:

$$\begin{aligned} J = {} & J_d + \mathrm{tr}\,[P(0)\hat{P}(0)] + \int_0^T [P^*\hat{P} + P\hat{Q}]dt \\ & + \lambda_\alpha^T(0)[\hat{\alpha}(0) - \alpha^\circ] + [\hat{\alpha}(0) - \alpha^\circ]^T P_\alpha(0)[\hat{\alpha}(0) - \alpha^\circ] \\ & + \mathrm{tr}\,\{[P_\alpha(0) + S_\alpha]\hat{P}_\alpha(0)\} \quad , \end{aligned} \tag{III-26}$$

where

$$S_\alpha \triangleq \int_0^T [\theta_{x\alpha}^T P_x \theta_{x\alpha} + \theta_{x\alpha}^T P_{x\alpha} \theta_\alpha + \theta_\alpha^T P_{x\alpha}^T \theta_{x\alpha} + \theta_\alpha^T P_\alpha \theta_\alpha] \mathrm{dt} \quad , \tag{III-27}$$

---

* θ is the fundamental matrix for the estimation equation linearized about the nominal trajectory.

and the vector λ_α, a part of the vector λ under a suitable partition, obeys Eq. (B-34).

The first term of J, J_d, is the deterministic cost, the second term is the cost caused by initial state uncertainty, the third is the cost caused by disturbances and measurement noise, and the remaining terms are the cost caused by parameter variations; hence, $P(0)$ may be interpreted as the sensitivity to initial state uncertainties, $P^*(t)$ as the sensitivity to uncertainty about the state of the plant because of measurement noise, and $P(t)$ as the sensitivity to disturbances. These latter three quantities were found in the previous work; the new quantities λ_α, P_α, and S_α are described in the sequel.

The first-order change in J caused by a change in nominal value can be computed by assuming temporarily that $\hat{\alpha}(0) \neq \alpha^\circ(0)$.* Because this performance increment is given by $\lambda_\alpha^T[\hat{\alpha}(0) - \alpha^\circ(0)]$, λ_α may be termed the sensitivity to the value of α. Consider Fig. III-1, in which the lower curve is the performance as a function of the actual value of a parameter when the controller optimal for that value is used and the upper curve is

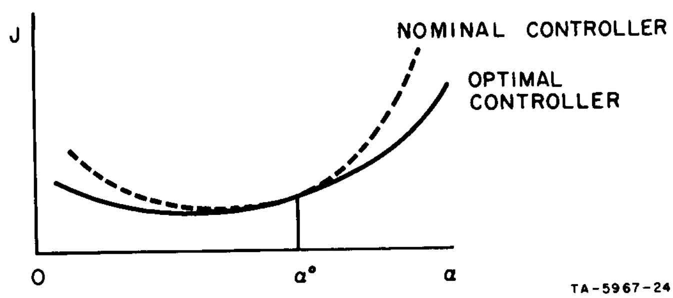

FIG. III-1 RELATIONSHIP BETWEEN PARAMETER SENSITIVITY QUANTITIES

the performance when the controller optimal for the nominal value is used; λ_α gives the slope of the two curves at the nominal value α_0. In system design, λ_α tells what parameters should be changed and in what direction to improve system performance.

---

*This contradicts the earlier assumption that $\hat{\alpha}(0) = \alpha^o(0)$, which was needed to show that the choice of K_α was irrelevant; however, λ_α is independent of K_α in any case and we are not concerned with second-order sensitivities here.

The term $\mathrm{tr}[P_\alpha(0)\hat{P}_\alpha(0)]$ gives the cost in performance resulting from differences between the actual and nominal values of α, even though the true value of α is determined at $t = 0$, while $\mathrm{tr}[S_\alpha \hat{P}_\alpha(0)]$ gives the additional cost when these differences are not determined; hence, $P_\alpha(0)$ is the sensitivity to *a priori* uncertainty in α while S_α is the sensitivity to not removing this uncertainty *a posteriori*. In Fig. III-1, P_α is the second derivative of the lower curve at a° and $P_\alpha + S_\alpha$ is the second derivative of the upper curve. P_α tells what parameters should have close tolerances, while S_α tells what parameters should be estimated or measured in an adaptive scheme to improve performance.

The value of λ, and thus λ_α, is independent of the choice of K and $\hat{K}$ [see Eq. (III-1)] and depends only upon u°. The independence of λ_α from K was first noted for linear systems with perfect measurement by Pagurek[19] and shown to hold for general systems by Witsenhausen[20]—a result to be expected in view of the above interpretation. P_α depends upon both u° and K but not on $\hat{K}$ whereas S_α depends on all three quantities; again those results are to be expected.

A program, described in Appendix A 3, has been developed to compute the sensitivity measures derived above as well as to perform the optimization presented in the next section. As presently coded, the program handles a general linear system with quadratic performance, but it can be extended to nonlinear systems and/or nonquadratic performances by writing subroutines for computation of the necessary partial derivatives.

5. Example

A program has been developed to compute the sensitivity measures defined above for a general linear system with quadratic performance. To illustrate the theory, a simple example solved by this program is presented. The plant under consideration, shown in Fig. III-2, is described by

$$\dot{x} = Fx + Gu + w \quad ,$$

$$z = Hx + v \quad , \qquad \text{(III-28)}$$

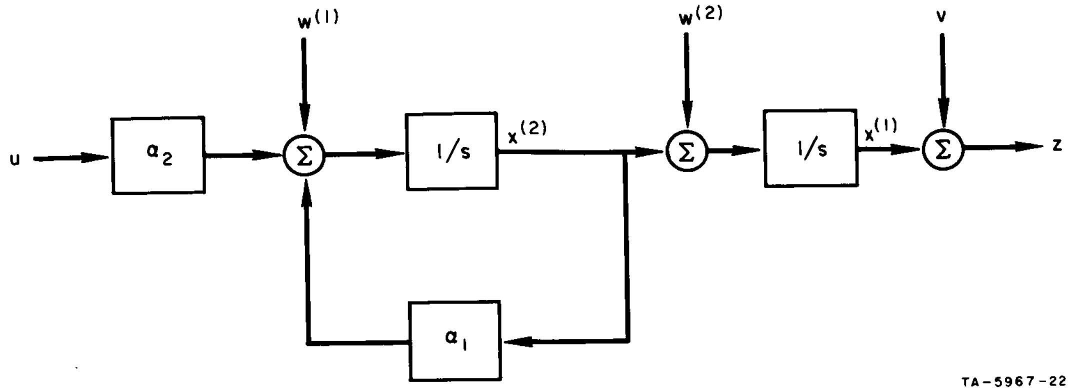

FIG. III-2 PLANT FOR ILLUSTRATIVE EXAMPLE

where

$$E[w(t)] = 0 \qquad E[w(t)w(\tau)^T] = \hat{Q}\delta(t-\tau)$$

$$E[v(t)] = 0 \qquad E[v(t)v(\tau)^T] = \hat{r}\delta(t-\tau)$$

$$F = \begin{bmatrix} 0 & 0 \\ 0 & \alpha^{(1)} \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \alpha^{(2)} \end{bmatrix}, \quad H = [1 \quad 0], \quad \hat{Q} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},$$

and $\hat{r}$ takes various values.

A gain parameter and a dynamic parameter with nominal values $\alpha^{(1)} = 1.0$ and $\alpha^{(2)} = -0.5$ respectively can vary. Initial conditions are $\hat{x}(0) = x^{\circ}(0)$, which takes various values, and

$$\hat{P}(0) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

and performance is defined by

$$J = r\int_0^1 u^2 dt + x^T(1)P(1)x(1) \quad , \qquad \text{(III-29)}$$

where $r = 0.03$ and $P(1)$ takes several values.

Tables III-1 to III-3 contain the sensitivity results for the illustrative example just described. In Table III-1, the quantities previously unspecified are given as well as the various parameter sensitivities, while Tables III-2 and III-3 present the nominal controls and trajectories as well as the disturbance and measurement noise sensitivities. Table III-1 also lists the performances J for the various situations; three values of J are given: the first corresponding to

$$\hat{P}_\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e., no parameter uncertainty, the second corresponding to

$$\hat{P}_\alpha = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

i.e., uncorrelated parameter uncertainty, and the third corresponding to

$$\hat{P}_\alpha = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$$

i.e., negatively correlated parameter uncertainty. In addition to the system with a noisy measurement as given in Eq. (III-28) results are given for a system with perfect measurement of x and no measurement.

Case 2 differs from Case 1 in having a different initial condition on x°, while Case 3 differs from Case 1 in having a different performance index. Note that the parameter sensitivities are a function of both initial condition and performance index, whereas the disturbance and measurement noise sensitivities depend only on the performance index because of the linearity of the system.

Consider Table III-1 in greater detail. The components of λ_α indicate that an improvement in performance can be obtained by increasing either the gain or the dynamic parameter but that a greater effect is obtained from changing the gain. Since the dynamic parameter is negative,

Table III-1

PARAMETER SENSITIVITY MEASURES FOR ILLUSTRATIVE EXAMPLE

CASE		1	2	3
$x^o(0)$		$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$P(1)$		$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\lambda_\alpha(0)$		$\begin{bmatrix} -0.0692 \\ -0.2023 \end{bmatrix}$	$\begin{bmatrix} -0.0735 \\ -0.6458 \end{bmatrix}$	$\begin{bmatrix} -0.0247 \\ -0.3531 \end{bmatrix}$
$P_\alpha(0)$		$\begin{bmatrix} 0.0118 & 0.0533 \\ 0.0533 & 0.2572 \end{bmatrix}$	$\begin{bmatrix} 0.0103 & 0.0399 \\ 0.0399 & 0.8212 \end{bmatrix}$	$\begin{bmatrix} 0.0188 & 0.0122 \\ 0.0122 & 0.3147 \end{bmatrix}$
Perfect Measurement	J	0.1385, 0.1697, 0.1567	0.3887, 0.4833, 0.4590	0.2960, 0.3437, 0.3480
	S_α	$\begin{bmatrix} 0.0090 & 0.0119 \\ 0.0119 & 0.0342 \end{bmatrix}$	$\begin{bmatrix} 0.0049 & 0.0070 \\ 0.0070 & 0.1091 \end{bmatrix}$	$\begin{bmatrix} 0.0130 & -0.0351 \\ -0.0351 & 0.1301 \end{bmatrix}$
Noisy Measurement	J	0.1847, 0.2489, 0.2137	0.4349, 0.6234, 0.5992	0.3728, 0.4504, 0.4344
	S_α	$\begin{bmatrix} 0.0483 & 0.1225 \\ 0.1225 & 0.3253 \end{bmatrix}$	$\begin{bmatrix} 0.0152 & 0.1038 \\ 0.1038 & 0.0388 \end{bmatrix}$	$\begin{bmatrix} 0.3472 & 0.0679 \\ 0.0679 & 0.0955 \end{bmatrix}$
No Measurement	J	0.2884, 0.3743, 0.3276	0.5385, 0.7914, 0.7564	0.4717, 0.5868, 0.5171
	S_α	$\begin{bmatrix} 0.0624 & 0.1803 \\ 0.1803 & 0.5274 \end{bmatrix}$	$\begin{bmatrix} 0.0134 & 0.1350 \\ 0.1350 & 1.6843 \end{bmatrix}$	$\begin{bmatrix} 0.4578 & 0.2862 \\ 0.2862 & 0.2593 \end{bmatrix}$

Table III-2

DISTURBANCE AND MEASUREMENT NOISE SENSITIVITIES FOR CASES 1 AND 2

t	CASE 1		CASE 2		$P(t)$	$P^*(t)$
	$x^o(t)$	$u^o(t)$	$x^o(t)$	$u^o(t)$		
0.0	$\begin{bmatrix}1.0000\\0.0000\end{bmatrix}$	-2.9926	$\begin{bmatrix}1.0000\\1.0000\end{bmatrix}$	-5.3472	$\begin{bmatrix}0.1141 & 0.0898\\0.0898 & 0.0706\end{bmatrix}$	$\begin{bmatrix}0.2687 & 0.2114\\0.2114 & 0.1663\end{bmatrix}$
0.1	$\begin{bmatrix}0.9857\\-0.2803\end{bmatrix}$	-2.7559	$\begin{bmatrix}1.0719\\0.4503\end{bmatrix}$	-4.9244	$\begin{bmatrix}0.1459 & 0.1057\\0.1057 & 0.0766\end{bmatrix}$	$\begin{bmatrix}0.3721 & 0.2697\\0.2697 & 0.1954\end{bmatrix}$
0.2	$\begin{bmatrix}0.9452\\-0.5233\end{bmatrix}$	-2.5070	$\begin{bmatrix}1.0923\\-0.0303\end{bmatrix}$	-4.4799	$\begin{bmatrix}0.1902 & 0.1253\\0.1253 & 0.0826\end{bmatrix}$	$\begin{bmatrix}0.5236 & 0.3452\\0.3452 & 0.2276\end{bmatrix}$
0.3	$\begin{bmatrix}0.8822\\-0.7296\end{bmatrix}$	-2.2455	$\begin{bmatrix}1.0681\\-0.4430\end{bmatrix}$	-4.0126	$\begin{bmatrix}0.2528 & 0.1493\\0.1493 & 0.0882\end{bmatrix}$	$\begin{bmatrix}0.7427 & 0.4387\\0.4387 & 0.2591\end{bmatrix}$
0.4	$\begin{bmatrix}0.8004\\-0.8996\end{bmatrix}$	-1.9706	$\begin{bmatrix}1.0059\\-0.7888\end{bmatrix}$	-3.5215	$\begin{bmatrix}0.3415 & 0.1770\\0.1770 & 0.0917\end{bmatrix}$	$\begin{bmatrix}1.0439 & 0.5411\\0.5411 & 0.2805\end{bmatrix}$
0.5	$\begin{bmatrix}0.7035\\-1.0339\end{bmatrix}$	-1.6817	$\begin{bmatrix}0.9125\\-1.0687\end{bmatrix}$	-3.0054	$\begin{bmatrix}0.4636 & 0.2051\\0.2051 & 0.0907\end{bmatrix}$	$\begin{bmatrix}1.4019 & 0.6203\\0.6203 & 0.2745\end{bmatrix}$
0.6	$\begin{bmatrix}0.5948\\-1.1326\end{bmatrix}$	-1.3781	$\begin{bmatrix}0.7944\\-1.2832\end{bmatrix}$	-2.4630	$\begin{bmatrix}0.6195 & 0.2246\\0.2246 & 0.0814\end{bmatrix}$	$\begin{bmatrix}1.6812 & 0.6097\\0.6097 & 0.2211\end{bmatrix}$
0.7	$\begin{bmatrix}0.4781\\-1.1963\end{bmatrix}$	-1.0592	$\begin{bmatrix}0.6580\\-1.4331\end{bmatrix}$	-1.8929	$\begin{bmatrix}0.7881 & 0.2196\\0.2196 & 0.0612\end{bmatrix}$	$\begin{bmatrix}1.6071 & 0.4479\\0.4479 & 0.1248\end{bmatrix}$
0.8	$\begin{bmatrix}0.3567\\-1.2249\end{bmatrix}$	-0.7238	$\begin{bmatrix}0.5099\\-1.5186\end{bmatrix}$	-1.2935	$\begin{bmatrix}0.9236 & 0.1758\\0.1758 & 0.0335\end{bmatrix}$	$\begin{bmatrix}1.0303 & 0.1962\\0.1962 & 0.0374\end{bmatrix}$
0.9	$\begin{bmatrix}0.2343\\-1.2185\end{bmatrix}$	-0.3711	$\begin{bmatrix}0.3564\\-1.5399\end{bmatrix}$	-0.6630	$\begin{bmatrix}0.9894 & 0.0965\\0.0965 & 0.0094\end{bmatrix}$	$\begin{bmatrix}0.3105 & 0.0303\\0.0303 & 0.0030\end{bmatrix}$
1.0	$\begin{bmatrix}0.1142\\-1.1772\end{bmatrix}$	0.0000	$\begin{bmatrix}0.2040\\-1.4971\end{bmatrix}$	0.0000	$\begin{bmatrix}1.0000 & 0.0000\\0.0000 & 0.0000\end{bmatrix}$	$\begin{bmatrix}0.0000 & 0.0000\\0.0000 & 0.0000\end{bmatrix}$

Table III-3

DISTURBANCE AND MEASUREMENT NOISE SENSITIVITIES FOR CASE 3

t	$x^o(t)$	$u^o(t)$	$P(t)$	$P^*(t)$
0.0	$\begin{bmatrix} 1.0000 \\ 0.0000 \end{bmatrix}$	-4.2642	$\begin{bmatrix} 0.2546 & 0.1279 \\ 0.1279 & 0.0810 \end{bmatrix}$	$\begin{bmatrix} 0.5455 & 0.3454 \\ 0.3454 & 0.2187 \end{bmatrix}$
0.1	$\begin{bmatrix} 0.9801 \\ -0.3841 \end{bmatrix}$	-3.6127	$\begin{bmatrix} 0.3162 & 0.1440 \\ 0.1440 & 0.0852 \end{bmatrix}$	$\begin{bmatrix} 0.6910 & 0.4089 \\ 0.4089 & 0.2420 \end{bmatrix}$
0.2	$\begin{bmatrix} 0.9259 \\ -0.6844 \end{bmatrix}$	-2.9279	$\begin{bmatrix} 0.3935 & 0.1604 \\ 0.1604 & 0.0887 \end{bmatrix}$	$\begin{bmatrix} 0.8578 & 0.4743 \\ 0.4743 & 0.2623 \end{bmatrix}$
0.3	$\begin{bmatrix} 0.8459 \\ -0.9014 \end{bmatrix}$	-2.2081	$\begin{bmatrix} 0.4878 & 0.1754 \\ 0.1754 & 0.0911 \end{bmatrix}$	$\begin{bmatrix} 1.0250 & 0.5324 \\ 0.5324 & 0.2765 \end{bmatrix}$
0.4	$\begin{bmatrix} 0.7484 \\ -1.0359 \end{bmatrix}$	-1.4514	$\begin{bmatrix} 0.5971 & 0.1856 \\ 0.1856 & 0.0921 \end{bmatrix}$	$\begin{bmatrix} 1.1484 & 0.5696 \\ 0.5696 & 0.2825 \end{bmatrix}$
0.5	$\begin{bmatrix} 0.6415 \\ -1.0881 \end{bmatrix}$	-0.6560	$\begin{bmatrix} 0.7139 & 0.1869 \\ 0.1869 & 0.0921 \end{bmatrix}$	$\begin{bmatrix} 1.1646 & 0.5739 \\ 0.5739 & 0.2828 \end{bmatrix}$
0.6	$\begin{bmatrix} 0.5335 \\ -1.0581 \end{bmatrix}$	0.1801	$\begin{bmatrix} 0.8247 & 0.1752 \\ 0.1752 & 0.0934 \end{bmatrix}$	$\begin{bmatrix} 1.0229 & 0.5455 \\ 0.5455 & 0.2909 \end{bmatrix}$
0.7	$\begin{bmatrix} 0.4326 \\ -0.9460 \end{bmatrix}$	1.0589	$\begin{bmatrix} 0.9136 & 0.1486 \\ 0.1486 & 0.1015 \end{bmatrix}$	$\begin{bmatrix} 0.7359 & 0.5028 \\ 0.5028 & 0.3436 \end{bmatrix}$
0.8	$\begin{bmatrix} 0.3470 \\ -0.7514 \end{bmatrix}$	1.9823	$\begin{bmatrix} 0.9702 & 0.1091 \\ 0.1091 & 0.1295 \end{bmatrix}$	$\begin{bmatrix} 0.3967 & 0.4710 \\ 0.4710 & 0.5591 \end{bmatrix}$
0.9	$\begin{bmatrix} 0.2850 \\ -0.4740 \end{bmatrix}$	2.9500	$\begin{bmatrix} 0.9953 & 0.0610 \\ 0.0610 & 0.2234 \end{bmatrix}$	$\begin{bmatrix} 0.1242 & 0.4545 \\ 0.4545 & 1.6635 \end{bmatrix}$
1.0	$\begin{bmatrix} 0.2549 \\ -0.1154 \end{bmatrix}$	3.8468	$\begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$	$\begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 33.3333 \end{bmatrix}$

increasing it implies reducing its absolute value. From the values of P_α and S_α it is seen that, for all three cases, the system is more sensitive to variations and uncertainties in gain than in the dynamic parameter. Because the control law is not optimized with respect to parameter variations the sensitivity is not necessarily a monotonic function of the quality of the measurement system. For example in Case 2 the sensitivity to dynamics changes is greater for the noisy measurement than no measurement and in Case 3 the system with noisy measurement leads to less sensitivity to gain changes than the system with perfect measurement.

Now consider Table III-2 and III-3. Case 3, which has more stringent performance requirements (*i.e.*, a cost on final velocity $x^{(2)}$ as well as position $x^{(1)}$) is more sensitive than Cases 1 and 2 to disturbances, velocity uncertainties, and position uncertainties occurring at early times. However, the latter cases are more sensitive than the former cases to position uncertainties occuring at later times.

D. Optimization

The controller used in the previous section consists of an optimal estimator of the state, followed by the optimal deterministic control law. Unfortunately, in many situations such a system is too sensitive to uncertainties and, therefore, the controller must be modified either to reduce the sensitivity (the stochastic effect) or to reduce the uncertainty (the dual control effect). Several options are available for reducing the sensitivity: By changing the nominal trajectory reduced sensitivity to uncertainties may be obtained at the cost of poorer deterministic performance. Changing K and $\hat{K}$ reduces the sensitivity to parameter variations and uncertainty at the expense of increased sensitivity to disturbances and measurement noise. The amount of uncertainty about parameter values can be reduced by estimating them (*i.e.*, using an adaptive controller), which for the perturbation situation under consideration means using the optimal value of $\hat{K}$ instead of $\hat{K}_\alpha = 0$ as in the previous section. Furthermore, because of the dual control effect, changing u° from the deterministic optimum may aid estimation of the parameters.

1. Problem Statement

Optimal values of u°, K, and $\hat{K}$ can be found by solving a deterministic control problem in which they are the control variables.

a. State Equations

The state vector for this control problem includes not only the state x° of the original problem, but also the adjoint variable λ, the cost matrices $\mathbf{P}$ and $\bar{\mathbf{P}}$, and the covariance $\hat{\mathbf{P}}$. Therefore, the state equations will include in addition to the state equation of the original problem, Eq. (III-5) in ordinary type; the adjoint equation, Eq. (III-13) in ordinary type, and Eqs. (III 14), (III-15), and (III-10) in bold-face type or their partitioned versions given in Appendix A-3.

For the nonadaptive optimization

$$\mathbf{K} = \begin{bmatrix} K \\ --- \\ K_\alpha \end{bmatrix} , \quad \hat{\mathbf{K}} = \begin{bmatrix} \hat{K} \\ --- \\ 0 \end{bmatrix} \tag{III-28}$$

where K and $\hat{K}$ are control variables to be determined along with u° and K_α is as given in Eq. (III-21). For adaptive optimization $\mathbf{K}$ is as given in Eq. (III-21) while

$$\hat{\mathbf{K}} = \hat{\mathbf{P}}\mathbf{H}^T\hat{R}^{-1}$$

$$= \begin{bmatrix} P_x H^T \hat{R}^{-1} \\ ----- \\ \hat{P}_{\alpha x} H^T \hat{R}^{-1} \end{bmatrix} \tag{III-29}$$

and only u° is to be determined.

In applying these formulas it should be remembered that $P = P_x$, but $\hat{P} \neq \hat{P}_x$.

b. Performance Index

The performance index to be minimized is found by substitution of the partitioned forms into Eq. (III-11) in bold-face type:

$$J = J_d + \mathrm{tr}[P(0)\hat{P}(0)] + \mathrm{tr}\{[P_\alpha(0) + S_\alpha]P_\alpha(0)\}$$

$$+ \int_0^T \mathrm{tr}(P\hat{Q} + P^*\hat{P} + 2A\bar{P}_x)dt \quad , \tag{III-30}$$

where

$$A = (\hat{K}\hat{R} - \hat{P}H^T)\hat{K}^T \quad ,$$

and

$$S_\alpha = \int_0^T \{\theta_{x\alpha}^T [P_x^* - H^T\hat{K}^T\overline{P}_x - (H^T\hat{K}^T\overline{P}_x)^T]\theta_{x\alpha}$$

$$+ \theta_{x\alpha}^T (P_{x\alpha}^* - H^T\hat{K}^T\overline{P}_{x\alpha})\theta_\alpha + \theta_\alpha^T (P_{x\alpha}^* - H^T\hat{K}^T\overline{P}_{x\alpha})^T\theta_{x\alpha}$$

$$+ \theta_\alpha^T P_\alpha^* \theta_\alpha\} dt \quad . \tag{III-31}$$

These results are essentially the same as those given in Eqs. (III-26) and (III-27) except for the additional terms containing $\overline{P}_x$ and $\overline{P}_{x\alpha}$, which are not zero in the nonadaptive case because K does not obey Eq. (III-21). Equations for $\overline{P}_x$ and $\overline{P}_{x\alpha}$ are given in Appendix A-3.

In performing the optimization a slightly different form of Eq. (III-30), where use of $\hat{P}$ and S_α is replaced by use of $\hat{P}_x$, $\hat{P}_{x\alpha}$, and $\hat{P}_\alpha$, is desirable:

$$J = J_d + \mathrm{tr}[P(0)\hat{P}_x(0)] + \mathrm{tr}[P_\alpha(0)\hat{P}_\alpha(0)]$$

$$+ \int_0^T [\mathrm{tr}(P\hat{Q} + P^*\hat{P}_x + 2A_x\overline{P}_x) + \mathrm{tr}(2P_{x\alpha}^{*T}\hat{P}_{x\alpha}^T + P_\alpha^*\hat{P}_\alpha + 2B_{\alpha x}\overline{P}_{x\alpha})] dt \quad , \tag{III-32}$$

where

$$A_x = (\hat{K}\hat{R} - \hat{P}_x H^T)\hat{K}^T$$

$$B_{\alpha x} = -(\hat{P}_{x\alpha}^T H^T)\hat{K}^T \quad .$$

Equations for $\hat{P}_x$, $\hat{P}_{x\alpha}$ and $\hat{P}_\alpha$ are given in Appendix A-3.

As the measurements become noiseless ($\hat{R}$ approaches zero), $\hat{K}$ becomes infinite; however, A_x and $B_{\alpha x}$ approach finite limits; for example, for perfect measurement of all components of x

$$A_x = 0$$

$$B_{\alpha x} = -\hat{P}_\alpha F_\alpha^T \qquad \text{(III-33)}$$

In the adaptive case with no measurement noise $\hat{P}_\alpha$ will, of course, be zero.

2. Problem Solution

The problem just described is a deterministic optimization problem that can be solved by use of the gradient method with the deterministic optimum values of u^o, K, and $\hat{K}$ as the initial choice of controls. To apply the gradient method, the Hamiltonion (which is a function of time; the above control variables*; the state variables† x^o, λ, P_x, $P_{x\alpha}$, P_α, $\bar{P}_x$, $\bar{P}_{x\alpha}$, $\hat{P}_x$, $\bar{P}_{x\alpha}$, and $\hat{P}_\alpha$; and their adjoints $\bar{\lambda}$, $\bar{x}$, Γ_x, $\Gamma_{\alpha x}$, Γ_α, $\bar{\Gamma}_x$, $\bar{\Gamma}_{\alpha x}$, $\hat{\Gamma}_x$, $\hat{\Gamma}_{\alpha x}$, $\hat{\Gamma}_\alpha$) must be written

$$\begin{aligned} H = \; & l(x^o, u^o, t) + \bar{\lambda}^T f(x^o, u^o, t) - \bar{x}^T(H_x^{oT} - K^T H_u^{oT}) \\ & + \mathrm{tr}(\hat{P}Q + P^*\hat{P}_x + 2A_x\bar{P}_x + \Gamma_x\dot{P}_x + 2\bar{\Gamma}_x\dot{\bar{P}}_x + \hat{\Gamma}_x\dot{\hat{P}}_x) \\ & + 2\,\mathrm{tr}(P^{*T}_{x\alpha}\hat{P}_{x\alpha} + B_{\alpha x}\bar{P}_{x\alpha} + \Gamma_{\alpha x}\dot{P}_{x\alpha} + \bar{\Gamma}_{\alpha x}\dot{\bar{P}}_{x\alpha} + \hat{\Gamma}_{\alpha x}\hat{P}_{x\alpha}) \\ & + \mathrm{tr}(P^*_\alpha\hat{P}_\alpha + \Gamma_\alpha\dot{P}_\alpha + \hat{\Gamma}_\alpha\hat{P}_\alpha) \qquad . \end{aligned} \qquad \text{(III-34)}$$

Note that the operation tr(AB) on matrices is equivalent to the operation $a^T b$ on vectors. By differentation of H with respect to the state variables, equations for the adjoint variables are obtained and by differentiation with respect to the controls the gradient is obtained. The results, which are complicated, are displayed in Appendix (A-3), which describes a program embodying this optimization technique.

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* Although the term *variable* is used, it should be understood the above quantities may be vectors or even matrices.

† Note that $\bar{P}_{\alpha x}$ and $\bar{P}_\alpha$ are zero because of the form of K_α used.

IV OPTIMUM DESIGN OF INSTRUMENTATION SYSTEMS

A. Introduction

The mathematical theory of deterministic optimal control is predicated upon the assumption that the complete state vector is perfectly known. This implies that the plant is fully instrumented and that the sensors are noiseless. In actual practice, it is usually uneconomic to measure the complete state vector and the assumption of perfectly noiseless sensors is illusory. Under these conditions, the designer attempts to achieve a degree of system performance sufficiently close to that of the idealized noiseless optimal control system with an instrumentation subsystem that can be economically justified and practically realized.

The most straightforward (though not always the best) approach to design feedback control systems with inaccurate and incomplete state information is to implement the optimum deterministic law of control found for the ideal system and to supply this law of control a suitable estimate of the state. As a consequence of sensor noise, these estimates are affected by errors and the performance is degraded since the control signal generated by the law of control does not match to the true state, but an estimate thereof.

During the design phase of the system, it is desirable to consider the trade-off between performance degradation caused by imperfect instrumentation and the dollar cost of the instrumentation subsystem.* In this section, a logical approach to select an instrumentation subsystem that minimizes performance degradation subject to restrictions on instrumentation cost (or bulk, weight, etc.) is presented.

* For reasons of convenience, we will characterize the instrumentation subsystem by its dollar cost. It is clear however, that weight or space constraints can be handled in exactly the same manner that cost constraints are handled here. Furthermore, reliability constraints, or even the R & D costs and risks associated with the procurement of a novel sensor can be handled by appropriate extensions of the procedure to be discussed. For details, the reader is referred to "The Application of Advanced Technologies for Future Missile Guidance Problems", by G. A. Branch, P. E. Merritt, J. Peschon, A. Korsak, Fourth Quarterly Report, Contract NOw 66-0364, SRI Project 5992, Stanford Research Institute, Menlo Park, California (April 1967) CONFIDENTIAL.

The step-by-step procedure to accomplish this design goal is as follows:

(1) Determine the optimum law of control assuming that the state is known perfectly. Let $u(x, t)$ denote this law of control.

(2) For an assumed sensor configuration, establish a law of estimation of the form $\hat{x}(x, u, v, t)$, where v is the sensor noise.

(3) Feed the state estimate $\hat{x}$ to the law of control $u(\cdot, t)$ and determine the performance degradation ΔJ caused by the sensor noise v. Relate ΔJ to a suitable statistical summary of v, characteristically its covariance matrix.

(4) Relate the cost C, in dollars, of the instrumentation subsystem to this same statistical summary of v; also establish a range from which this instrumentation subsystem can be selected.

(5) Select from this range that instrumentation subsystem which minimizes the performance degradation ΔJ subject to an upper bound on cost C.

In this formulation, the instrumentation subsystem is treated as a resource, which should be used in the best possible manner. As the implementation of most instrumentation systems (especially in space applications, where instrumentation usually includes telemetry and as well entails significant expenditures of cost, weight, bulk, etc.) it is of paramount importance to select its characteristics carefully in relation to the systems performance function.

Note: For an instrumentation subsystem of given quality, the performance obtained in this manner is usually not optimum, even if the estimator selected in Step (2) is an optimum (for example, least variance) estimator. The reasons for this is that the rigorous approach to design control systems with noisy state information is the theory of combined optimum control and estimation;[5] one practical consequence of this theory is that the law of control in general not only depends on the estimate $\hat{x}$, but also on the higher moments of this estimate. However, the design procedure previously outlined can be extended in principle to the combined optimum control and estimation problem, since the cost J ultimately depends on the characteristics of the instrumentation subsystem. The approximate solution of combined optimization problems discussed in Sec. III provides this functional relation.

B. Control System Configurations

We consider a closed-loop control system having the general configuration shown in Fig. IV-1.

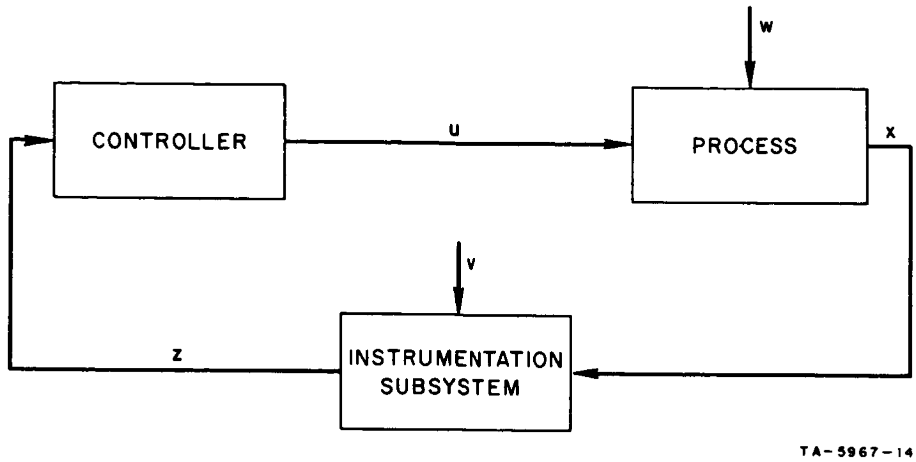

FIG. IV-1 CONTROL SYSTEM, COMPRISING A PROCESS, AN INSTRUMENTATION SUBSYSTEM, AND A CONTROLLER

Given the process, the perturbations $w(t)$ and $v(t)$ and the initial condition of state x_0, the operating cost

$$J = E\{\int_0^T l(x,u,t)dt + \Phi[x(T)]\} \tag{IV-1}$$

depends on the characteristics of the controller and of the instrumentation subsystem.

The characteristics of the controller can be selected on the basis of one of the following assumptions:

1. Nonoptimal Control and Filtering

This is the traditional approach for servo system synthesis. The controller embodies two relations of the form

$$u = g(\hat{x}, t) \tag{IV-2}$$

$$\dot{\hat{x}} = \hat{f}(\hat{x}, u, v, t) \quad , \tag{IV-3}$$

Equation (IV-2) represents the law of control, whereas Eq. (IV-3) represents the law of estimation. After both laws have been selected, possibly on the basis of computer simulations, the closed-loop system of Fig. IV-1 can be described by a vector differential equation of the form

$$\dot{\bar{x}} = f(\bar{x}, w, v, t) \quad , \qquad \text{(IV-4)}$$

where

$$\bar{x} = \begin{bmatrix} x \\ \\ \hat{x} \end{bmatrix}$$

represents the expanded state of the system; this state includes the plant and estimator (filter) dynamics. Its performance is measured by the cost function of Eq. (IV-1), in which the expectation is taken over the perturbations v and w.

2. Optimal Deterministic Control

It is now assumed that the process state x is known with perfect accuracy and that there are no perturbations, *i.e.*, $w = 0$. The classical theory of deterministic optimum control yields, for a closed-loop system configuration such as shown in Fig. IV-1, a law of control

$$u = g(x, t) \quad , \qquad \text{(IV-5)}$$

which would minimize the deterministic cost

$$J_d = \int_0^T l(x, u, t)dt + \Phi[x(T)] \quad , \qquad \text{(IV-6)}$$

if x were known perfectly and w were zero.

If every component x_i of x is measured with additive noise v_i, the control signal generated by the law (IV-5) would actually be

$$u = g(x + v, t) \quad . \qquad \text{(IV-7)}$$

The closed-loop system would again be described by a vector differential equation of the form

$$\dot{x} = f(x, w, v, t) \qquad \text{(IV-8)}$$

and the true operating cost J would be measured by the expression given in Eq. (IV-1).

3. Optimal Estimation

Instead of feeding the raw state measurements to the optimum deterministic controller, as was done in Eq. (IV-7), one may assume that an optimum estimation scheme is used to filter the measurements z. It is well known from the theory of linear estimation that the optimum estimator output is described by a vector differential equation of the form

$$\dot{\hat{x}} = \hat{f}(\hat{x}, x, v, t) \qquad \text{(IV-9)}$$

and it can be proven that the output of a nonlinear estimator would be described by an equation of the same form. Consequently, the motion of the closed-loop system comprising an optimum deterministic controller connected to an optimum estimator would again be given by a vector differential equation of the form

$$\dot{\bar{x}} = f(\bar{x}, w, v, t) \quad , \qquad \text{(IV-10)}$$

where $\bar{x}$ comprises the plant and estimator states x and $\hat{x}$.

One may approach the design of the controller/estimator pair using increasingly realistic assumptions culminating in an optimum combined control and estimation scheme as discussed in Sec. III, but in all cases the operating cost given by Eq.(IV-1) will depend on the statistics of the perturbations w and v.

In this section, we concern ourselves primarily with the perturbations v generated by the instrumentation subsystem, the properties of which are summarized by a vector s. This vector fully describes the relevant performance characteristics of the measurement subsystem; that is, the statistics of v. In most practical applications, the components s, s_i of the vector s would be the diagonal elements σ_{ii} of the measurement noise covariance matrix $\hat{R}$.

Note: The vector s may also characterize the topology of the instrumentation subsystem in the following manner: $s_i = \infty$ may mean that sensor i has been removed from the instrumentation subsystem. This is a perfectly acceptable limiting process for the Kalman estimator in which the absence of the ith instrument is accounted for by making the element σ^2_{ii} of R equal to infinity. For other types of estimators, this may not be a permissible interpretation.

To summarize, we may state in a general way that there exists a functional relation

$$J(s) \tag{IV-11}$$

between the operating cost of the system and the vector s characterizing the performance of the instrumentation subsystem. This functional relation depends on the way the sensor outputs z are used in the controller/estimator block, but for given laws of control and estimation, J is uniquely related to s.

C. Optimum Selection of the Instrumentation Subsystem

The ideal instrumentation subsystem is one which allows the estimator to produce a perfectly accurate estimate of state; this usually implies that the measurements must be noise free and that every state variable of the plant be observable. Even if it were possible to perform perfectly noise free measurements, the dollar cost of the instrumentation subsystem might not be justified in relation to the operating cost $J(s)$ it would bring about.

A very practical problem of system design optimization is therefore to minimize $J(s)$ subject to an upper bound $\overline{C}$ on the dollar cost one is willing to allocate to the instrumentation subsystem. Since the vector s completely characterizes the instrumentation subsystem, the dollar cost C is a function of s, $C(s)$.

The optimization problem is thus summarized as

$$\min_s J(s) \quad , \tag{IV-12}$$

subject to

$$C(s) \leq \overline{C} \quad . \tag{IV-13}$$

For technical reasons, or reasons of availability, the parameter vector s_i may be bounded; that is,

$$s \in S \quad . \tag{IV-14}$$

Also, in actual design practice, one ordinarily does not have an infinite range from which s can be selected; rather, there exists on the market a finite number of instrument makes, each of which is characterized by a fixed s_i.

The optimization problem of Eqs. (IV-12), (IV-13), and (IV-14) is a standard nonlinear programming problem, which in principle can be solved in a number of efficient ways. It is worthwhile to note that the problem of optimally selecting the measurement subsystem has been reduced to the standard format of the resource allocation problems customarily solved by nonlinear programming techniques.

D. Special Case

1. Discussion

It is assumed that the general configuration of the instrumentation subsystem has been selected and that the dollar cost $C(s)$ can be expressed as a sum of individual sensor costs

$$C(s) = \sum_i C_i(s_i) \qquad i = 1, \ldots, n \quad , \tag{IV-15}$$

where the continuous variable s_i is the noise power of the ith sensor. Since in well-behaved problems, the operating cost $J(s)$ is decreased when the instrumentation subsystem cost C is increased, the solution will be such that

$$\sum_i C_i(s_i) = \overline{C} \quad . \tag{IV-16}$$

We further assume that s_i cannot be made smaller than $\underline{s}_i$, the minimum noise power obtainable for this particular class of sensors.

Under these assumptions and if the functions $J(s)$ and $C_i(s_i)$ meet the required conditions of smoothness and convexity, the optimum solution is given by Kuhn and Tucker conditions[21] which require that the derivatives of the function

$$J(s) + \lambda[\sum_i C_i(s_i) - \overline{C}] + \sum_i \mu_i(s_i - \underline{s}_i)$$

with respect to the variables s_i be zero. The following necessary conditions of optimality are thus obtained:

$$\frac{\partial J}{\partial s_i} + \lambda \frac{\partial C_i}{\partial s_i} - \mu_i = 0 \tag{IV-17}$$

$$\sum_i C_i(s_i) = \overline{C} \tag{IV-18}$$

$$\mu_i(\underline{s}_i - s_i) = 0 \qquad i = 1, \ldots, n \quad , \tag{IV-19}$$

where λ and μ_i are the dual variables associated with the constraints $\overline{C}$ and $s_i \geq \underline{s}_i$.

If the optimum solution is unconstrained for two or more sensors i, j; that is, $s_i > \underline{s}_i$ and $s_j > \underline{s}_j$, then

$$\mu_i = \mu_j = 0 \tag{IV-20}$$

and

$$\frac{\dfrac{\partial J}{\partial s_i}}{\dfrac{\partial C_i}{\partial s_i}} = \frac{\dfrac{\partial J}{\partial s_j}}{\dfrac{\partial C_j}{\partial s_j}} = \text{constant} = -\lambda \ . \tag{IV-21}$$

The economic interpretation of Eq. (IV-21) is that the marginal decrease in operating cost $-(\partial J/\partial C_i)$ per additional dollar spent in reducing s_i should be the same for all unconstrained sensors.

Note: The dual variables λ and μ_i contain the following sensitivity information:

$$\lambda = -\frac{\Delta J}{\Delta \overline{C}} \tag{IV-22}$$

$$\mu_i = \frac{\Delta J}{\Delta \underline{s}_i} \ ; \tag{IV-23}$$

that is, λ indicates how the operating cost J changes with the instrumentation budget $\overline{C}$ assuming that this budget is used optimally as previously defined; μ_i indicates how the operating cost would vary if a higher-quality instrument having a lower limit $\underline{s}_i$ could be found or developed.

2. Numerical Example

The linear second-order plant shown in Fig. IV-2 is controlled by an optimum sampled data controller/estimator designed to minimize the cost function

$$J = E\{0.01 \sum_{k=0}^{8} u_k^2 + [x_9^{(1)}]^2\} \ . \tag{IV-24}$$

The initial state is [0,0] and the noise powers of the measurements of $x^{(1)}$ and $x^{(2)}$ are σ_1^2 and σ_2^2, respectively. The sampling time is 1 second.

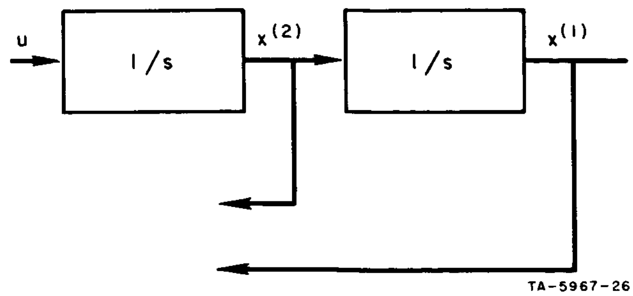

FIG. IV-2 SECOND-ORDER LINEAR PLANT WITH NOISY MEASUREMENTS OF THE STATE VARIABLES $x^{(1)}$, $x^{(2)}$

The operating cost J of this system is the sum of a deterministic cost J_d (which is zero here) and a stochastic cost J_s, the magnitude of which is a function of the measurement subsystem parameters

$$s_1 = \sigma_1^2$$

$$s_2 = \sigma_2^2 \quad . \tag{IV-25}$$

The stochastic cost $J_s(s_1, s_2)$ was computed for a range of reasonable values of s_1 and s_2 with a program implementing the optimum laws of linear control and estimation.* This functional relation was thereafter approximated by fitting the following quadratic expression to the points perviously computed

$$J_s \cong 1.216\, s_1 + 2.632\, s_2 - 1.185\, s_1^2 - 2.392\, s_2^2 + 0.431\, s_1 s_2 \quad . \tag{IV-26}$$

Curves of constant J_s computed from Eq. (IV-26) are shown in the s_1, s_2 plane in Fig. IV-3.

Over the range s_1, s_2 of interest, the dollar cost $C_1(s_1)$ and $C_2(s_2)$ of the $x^{(1)}$, $x^{(2)}$ sensors are assumed to be

$$C_1(s_1) = \frac{1}{s_1}$$

$$C_2(s_2) = \frac{0.543}{s_2} \quad . \tag{IV-27}$$

* For a detailed summary of these laws, see Sec. III and Ref. 5.

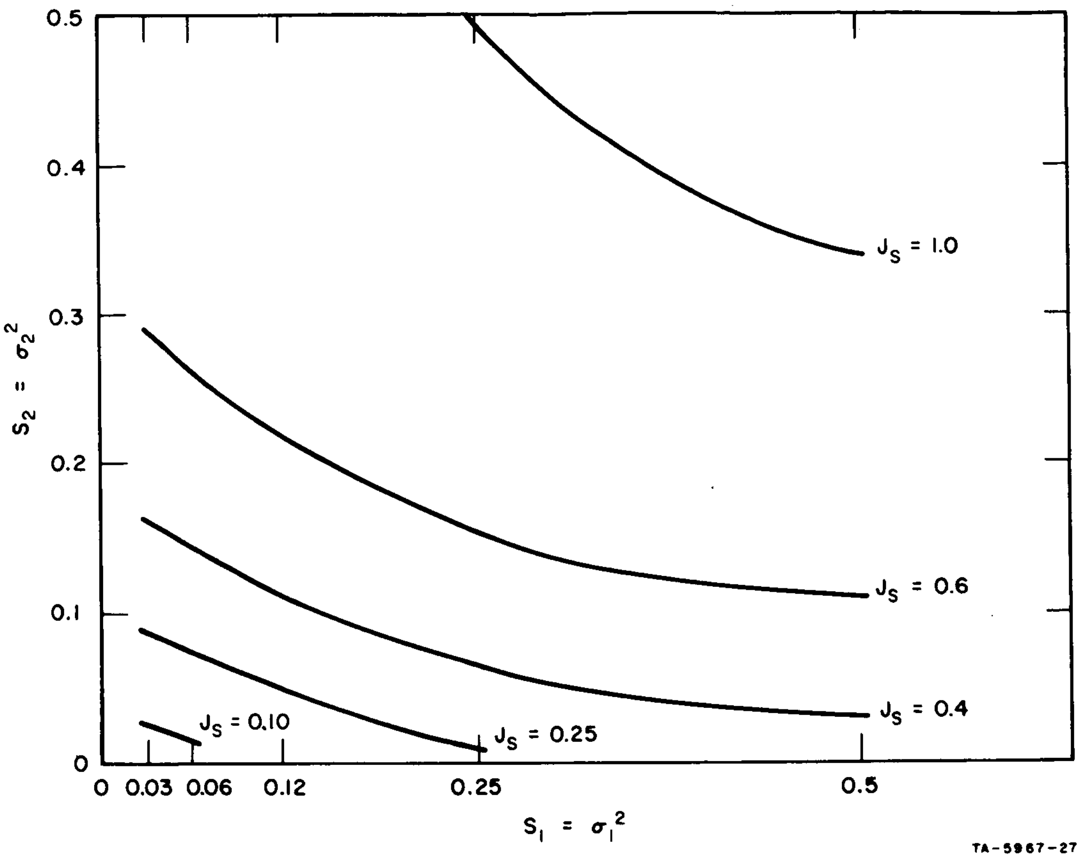

FIG. IV-3 CURVES OF CONSTANT STOCHASTIC COST J_s IN THE s_1, s_2 PLANE

An upper bound $\overline{C}$ = 9.43 monetary units has been imposed. With given lower bounds $\underline{s}_1$, $\underline{s}_2$ of the available sensors, the optimum design parameters s_1, s_2 of the measurement subsystem are obtained from Eqs. (IV-17) through (IV-19), which for the numerical example of interest, become:

$$1.216 - 2.370\, s_1 + 0.431\, s_2 - \lambda \frac{1}{s_1^2} - \mu_1 = 0 \quad . \tag{IV-28}$$

$$2.632 + 0.431\, s_1 - 4.784\, s_2 - \lambda \frac{0.543}{s_2^2} - \mu_2 = 0 \quad , \tag{IV-29}$$

$$\frac{1}{s_1} + \frac{0.543}{s_2} = 9.43 \quad , \tag{IV-30}$$

$$\mu_1(\underline{s}_1 - s_1) = 0 \quad ,$$

$$\mu_2(\underline{s}_2 - s_2) = 0 \quad . \tag{IV-31}$$

Assuming first that the constraints $\underline{s}_1$, $\underline{s}_2$ are sufficiently low, the dual variables μ_1 and μ_2 are zero and the unknowns s_1, s_2, and λ are given by Eqs. (IV-28), (IV-29), and (IV-30) as

$$
\begin{aligned}
s_1 &= 0.250 \\
s_2 &= 0.100 \\
\lambda &= 0.0417 \quad . \qquad \text{(IV-32)}
\end{aligned}
$$

The stochastic cost $J_s(0.250, 0.100)$ is 0.4806. The dual variable λ carries the sensitivity information

$$\lambda = -\frac{\Delta J_s}{\Delta \overline{C}} = 0.0417 \quad .$$

If the upper bound on cost $\overline{C}$ were changed from 9.43 to 10.43, the stochastic cost would decrease by approximately 0.0417.

We next assume that the most accurate $x^{(1)}$-sensor available has a noise power of $\underline{s}_1 = 1/3 = 0.333$. The design parameter s_2 is now given directly by Eq. (IV-30) and the dual variable λ and $\mu_1 \neq 0$ are obtained from the pair (IV-28), (IV-29) as

$$
\begin{aligned}
s_2 &= 0.0845 \\
\lambda &= 0.0311 \\
\mu_1 &= 0.1825 \quad . \qquad \text{(IV-33)}
\end{aligned}
$$

The dual variable μ_2 of course is zero, since s_2 is not constrained.

Note that λ has decreased substantially, which means that the reduction of operating cost per instrumentation dollar spent is now much less. The stochastic cost $J_s(0.333, 0.0845)$ has increased to 0.4908. The dual variable μ_1 measures the sensitivity of J_s with respect to a relaxation $\Delta\underline{s}_1$ in the imposed lower bound; if $\underline{s}_1$ were reduced from 0.333 to 0.250 (the previous unconstrained case), the predicted cost reduction would be

$$\Delta J_s = 0.1825 \cdot 0.833 = 0.0152 \quad ,$$

whereas the actual cost reduction is only 0.0102. The reason for the lack of accuracy in predicting ΔJ_s is the large increment Δs_1 chosen to illustrate the technique.

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V OPTIMAL CONTROL OF MEASUREMENT SUBSYSTEMS

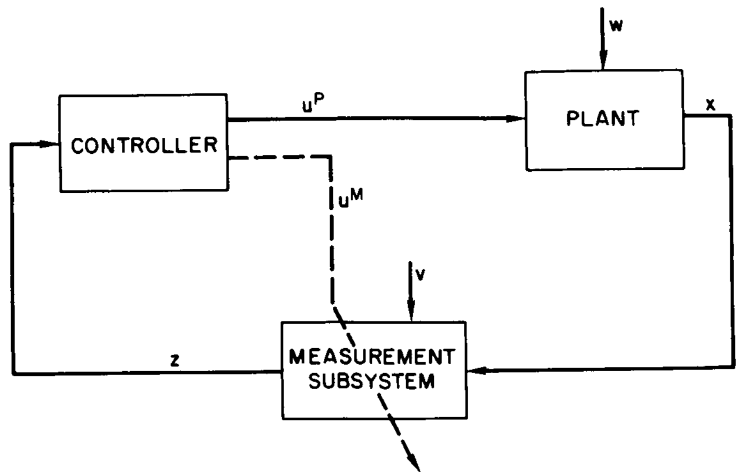
A. Introduction

This section considers an important class of problems known as measurement adaptive problems, in which control is available over not only the plant (*i.e.*, the state equation contains a control variable) but also the measurement subsystem (*i.e.*, the measurement equation contains a control variable). In the general situation the problem is shown to be a generalization of the combined optimization problem. In the special situation of linear systems, quadratic cost, and Gaussian random processes, it is shown that the optimization of plant control can be carried out independently of the measurement control optimization and, furthermore, that optimization of the measurement control can be done *a priori*. Two examples illustrating this latter situation are presented.

The adaptive systems commonly discussed in the literature are designed to compensate for both uncertainty about the plant and environmental changes by altering the control signals supplied to the plant. In this paper a different class of adaptive systems characterized by the presence of control action upon the measurement subsystem is discussed. A paper by Athans and Schweppe[22] considers the design of an optimum modulating signal in an estimation problem. The present paper is more general than that work in that it treats the general control of the measurement subsystem within a feedback control system.

The systems under consideration, referred to as *measurement adaptive systems*, take the general form shown in Fig. V-1; the unique feature of this block diagram is the control signal u^M supplied by the controller to the measurement subsystem. Feldbaum[23] has noted that a plant control has the dual purpose of taking the plant to a desired state and obtaining information about the actual state of the plant; the measurement subsystem control, on the other hand, serves only informational purposes.

The problem under discussion is representative of an important class of optimal decision processes not directly covered by the classical theory of optimal control. In the remainder, a mathematical formulation of the



x = PLANT STATE VECTOR
w = RANDOM PERTURBATION VECTOR
z = MEASUREMENT VECTOR
v = MEASUREMENT NOISE VECTOR
u^P = CONTROL VECTOR SUPPLIED TO THE PLANT
u^M = CONTROL VECTOR SUPPLIED TO THE MEASUREMENT SUBSYSTEM

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FIG. V-1 MEASUREMENT ADAPTIVE SYSTEM

general problem will be provided and a solution derived from combined optimization theory[5,24] will be developed. Thereafter the computable and practically important special case of a linear system with Gaussian perturbations and quadratic performance will be treated in detail.

It will be seen that in this case the optimum measurement u^M can be determined *a priori* by solution of a deterministic control problem in which the elements of the covariance matrix of the state estimate enter as state variables.

Some examples of measurement adaptive problems include: finding the optimum channel allocation among the various components of a measurement vector when they must be transmitted over a time-shared communication channel of limited bandwidth, finding the optimum timing of measurements when the number of possible measurements is limited because of energy constraints, and finding the optimum trade-off between measurement of range and range rate in a radar system with given ambiguity function.

Note that the decisions required in the above examples are dynamic in nature; *i.e.*, the optimum trade-off between velocity and position measurement will generally vary in time, as a simplified version of the radar example presented in the sequel shows.

A second large class of measurement adaptive problems arises when the constraints on the measurement control are replaced by the cost of making measurements. For example, in anti-submarine warfare, there is a cost of alerting the enemy submarine every time an active sonar signal is sent to measure its position and velocity. Examples abound of operational-type problems in which it costs dollars and cents to make measurements. In these problems the cost incurred by making measurements is added to the customary cost of operating the system and it is desired to find the optimum balance between the cost of measurement and the saving in performance costs made possible by the measurements.

B. General Problem Formulation

In this section the general measurement adaptive problem is considered. In the presentation it is convenient to use the symbol Z_k for the time history of a quantity z_i; *i.e.*,

$$Z_k \triangleq (z_0, \ldots, z_k) \quad . \tag{V-1}$$

The symbol z_{-1} is used to represent the empty sequence.

1. Statement of the Problem

In the general case, the problem of measurement system adaptation is formulated as follows:

Given:
the plant equation, written in discrete time as

$$x_{k+1} = f_k(x_k, u_k^P, w_k) \qquad k = 0, \ldots, N \quad ; \tag{V-2}$$

the measurement equation,

$$z_k = h_k(x_k, u_k^M, v_k) \qquad k = 0, \ldots, N \quad ; \tag{V-3}$$

the probability densities

$$p(x_0),\ p(w_k),\ p(v_k) \qquad k = 0, \ldots, N \quad , \tag{V-4}$$

where the random variables w_k and v_k, which are white, and x_0 are independent of each other; and the performance criterion (cost function)

$$J = \psi(u_0^M) + E\left\{\sum_{k=0}^{N-1} l_k(x_k, u_k^P, u_{k+1}^M) + l_N(x_N, u_N^P) + \phi(x_{N+1})\right\} , \tag{V-5}$$

where the expectation is taken with respect to the random variable x_k.

Find:

the controls $u_k^P(Z_k)$ of the plant and $u_k^M(Z_{k-1})$ of the measurement subsystem ($k = 0, \ldots, N$) that minimize the performance criterion J, subject to the constraints that

$$u_k^P \in \mathcal{U}_k^P , \tag{V-6}$$

$$u_k^M \in \mathcal{U}_k^M , \tag{V-7}$$

$$y_{N+1} \in \mathcal{Y} , \tag{V-8}$$

where

$$y_0 = g_{-1}(u_0^M)$$

$$y_{k+1} = g_k(y_k, u_k^P, u_{k+1}^M) \qquad k = 0, \ldots, N-1$$

$$y_{N+1} = g_N(y_N, u_N^P) .$$

Several comments on this problem formulation are in order. The form chosen for J was selected because it is the most general form that can be handled by the dynamic programming technique to be described. The purpose of the vector y_k is to allow most, if not all, global constraints to be converted into local constraints. An example of this procedure is given later.

2. Solution

The first step in the solution consists of defining

$$u_k = \begin{bmatrix} u_k^P \\ \\ u_{k+1}^M \end{bmatrix}, \quad \mathscr{u}_k = \begin{bmatrix} \mathscr{u}_k^P \\ \\ \mathscr{u}_{k+1}^M \end{bmatrix} \quad k = 0, \ldots, N-1 . \qquad \text{(V-9)}$$

In terms of u_k the measurement adaptive problem is very similar to the combined optimization problem treated in Refs. 5 and 24, the major difference being that the control does not affect the measurement equation in the combined optimization problem. For this reason, the solution is only outlined and details of the proof omitted. Because it summarizes all information about the plant state x_k, the key quantity is the information state $\mathscr{p}_k$, defined by

$$\mathscr{p}_k \triangleq p(x_k/Z_k, U_{k-1}, u_0^M) \qquad \text{(V-10)}$$

for the measurement adaptive problem. A recursive equation for the information state of the form

$$\mathscr{p}_{k+1} = F_k(\mathscr{p}_k, u_k, z_{k+1}) \quad k = 0, \ldots, N-1 \qquad \text{(V-11)}$$

may be found by application of Bayes rule;[16] the result is shown in Eq. (V-12) below. The probability densities $p(x_{k+1}/x_k, u_k^P)$ and $p(z_k/x_k, u_k^M)$ can be obtained from Eqs. (V-2), (V-3) and (V-4).

$$p(x_{k+1}/Z_{k+1}, U_k, u_0^M) = \frac{p(z_{k+1}/x_{k+1}, u_{k+1}^M)\int_{x_k} p(x_{k+1}/x_k, u_k^P)p(x_k/Z_k, U_{k-1}, u_0)dx_k}{\int_{x_{k+1}} p(z_{k+1}/x_{k+1}, u_{k+1}^M)\int_{x_k} p(x_{k+1}/x_k, u_k^P)p(x_k/Z_k, U_{k-1}, u_0^M)dx_k dx_{k+1}}$$

$$k = 0, \ldots, N-1$$

$$p(x_0/z_0, u_0^M) = \frac{p(z_0/x_0, u_0^M)p(x_0)}{\int_{x_0} p(z_0/x_0, u_0^M)p(x_0)dx_0} \qquad \text{(V-12)}$$

In terms of $\wp_k$, a dynamic programming algorithm can be derived for solution of the measurement adaptive problem. First the performance criterion is rewritten in terms of $\wp_k$:

$$J = \psi(u_0^M) + E\left\{\sum_{k=1}^{N-1} L_k(\wp_k, u_k) + \Phi(\wp_N, u_N^P)\right\} , \tag{V-13}$$

where

$$L_k(\wp_k, u_k) = \int_{x_k} l_k(x_k, u_k^P, u_{k+1}^M) p(x_k/Z_k, U_{k-1}, u_0^M) dx_k$$

$$k = 0, \ldots, N-1$$

$$\Phi(\wp_N, u_N^P) = \int_{x_N} [l_N(x_N, u_N^P) + \int_{w_N} \phi[f_N(x_N, u_N^P, w_N)] p(w_N) dw_N]$$

$$\cdot\ p(x_N/Z_N, U_{N-1}, u_0^M) dx_N \quad .$$

Application of the principal of optimality to Eqs. (V-11) and (V-13) yields a recursive equation for the return function $I_k(\wp_k, y_k)$

$$I_k(\wp_k, y_k) = \min_{u_k \in \mathcal{U}_k} \left(L_k(\wp_k, u_k) + \underset{z_{k+1}}{E} \{I_{k+1}[F_k(\wp_k, u_k, z_{k+1}), g_k(y_k, u_k)]\}\right)$$

$$k = 0, \ldots, N-1 \quad ,$$

$$I_N(\wp_N, y_N) = \min_{u_N^P \in \mathcal{U}_N^P} \Phi(\wp_N, N) \tag{V-14}$$

$$y_{N+1} \in \mathcal{Y}$$

Finally, since $\wp_0$ is a function of u_0^M [see Eq. (V-12)]:

$$J = \min_{u_0^M \in \mathcal{U}_0^M} \{\psi(u_0^M) + I_0(\wp_0\ y_0)\} \quad . \tag{V-15}$$

C. Special Case

Because in the general case the information state $\wp_k$ is infinite-dimensional, solution of the measurement adaptive problem is not practical without some sort of approximation. However, if the plant is linear, if the measurement subsystem is linear in the state and measurement noise (but not necessarily the measurement control), if the disturbance and measurement noise are Gaussian, and if the performance is quadratic in the state and plant control with an additive measurement control cost term, then the plant control policy can be determined separately from the measurement control policy, which is open-loop (that is, the optimum measurements may be determined *a priori*). This special case is the topic of the following section.

1. Statement of the Problem

In the special case: the plant equation is

$$x_{k+1} = F_k x_k + G_k u_k^P + w_k \quad . \tag{V-16}$$

The measurement equation is

$$z_k = H_k(u_k^M) x_k + v_k \quad , \tag{V-17}$$

where $H_k(u_k^M)$ gives the relationship between the measurement matrix and the measurement control. The probability density functions are

$$p(x_0) = c_1 \exp\left\{-\frac{1}{2}[(x_0 - \bar{x}_0)^T(\hat{P}_{0/-1})^{-1}(x_0 - \bar{x}_0)]\right\}$$

$$p(w_k) = c_2 \exp\left\{-\frac{1}{2}[w_k^T \hat{Q}_k^{-1} w_k]\right\}$$

$$p(v_k) = c_3 \exp\left\{-\frac{1}{2}[v_k^T \hat{R}_k^{-1}(u_k^M) v_k]\right\} \quad , \tag{V-18}$$

where $\hat{R}_1(u_k^M)$ gives the relationship between the measurement noise and the measurement control, and c_1, c_2, c_3 are constants of no consequence here. The performance criterion is

$$J = E\left\{\sum_{k=0}^{N} [x_k^T Q_k x_k + u_k^{P^T} R_k u_k^P + l_k^M(u_k^M)] + x_{N+1}^T P_{N+1} x_{N+1}\right\} \quad . \tag{V-19}$$

The plant control u_k^P is unconstrained; the constraints on the measurement control u_k^M are given by Eqs. (V-7) and (V-8), with g_k independent of u_k^P and $y_{N+1} \equiv y_N$.

2. Solution of the Problem

If the u_k^M were specified, then the above problem would reduce to the linear combined optimization problem, whose solution is presented in Refs. 5, 25 and 26. The optimal control in that case is

$$u_k^P = -K_k \hat{x}_{k/k} \quad , \tag{V-20}$$

where

$$K_k = (G_k^T P_{k+1} G_k + R_k)^{-1} G_k^T P_{k+1} F_k$$

$$P_k = Q_k + F_k^T P_{k+1} F_k - P_{k+1}^*$$

$$P_{k+1}^* = F_k^T P_{k+1} G_k (G_k^T P_{k+1} G_k + R_k)^{-1} G_k^T P_{k+1} F_k$$

$$k = N, \ldots, 0 \quad ; \tag{V-21}$$

and $\hat{x}_{k/k}$, the optimal estimate of x_k conditioned on Z_k, is given by

$$\hat{x}_{k/k} = F_{k-1}\hat{x}_{k-1/k-1} + G_{k-1}u_{k-1}^P + \hat{K}_k [z_k - H_k(F_{k-1}\hat{x}_{k-1/k-1} + G_{k-1}u_{k-1}^P)] \quad , \tag{V-22}$$

where

$$\hat{K}_k = \hat{P}_{k/k-1} H_k^T (H_k \hat{P}_{k/k-1} H_k^T + \hat{R}_k)^{-1}$$

and $\hat{P}_{k/k}$, the conditional covariance of the error in the estimate of x_k given Z_k, satisfies

$$\hat{P}_{k/k} = \hat{P}_{k/k} - \hat{P}_{k/k-1} H_k^T (H_k \hat{P}_{k/k-1} H_k^T + \hat{R}_k)^{-1} H_k \hat{P}_{k/k-1}$$

$$\hat{P}_{k/k-1} = \hat{Q}_{k-1} + F_{k-1} \hat{P}_{k-1/k-1} F_{k-1}^T$$

$$k = 0, \ldots, N \quad . \tag{V-23}$$

The optimal performance is (see the Appendix or Ref. 5 for derivation):

$$J = \bar{x}_0^T P_0 \bar{x}_0 + \mathrm{tr}\,[P_0 \hat{P}_{0/-1}] + \sum_{n=0}^{N} \Delta\beta_k \quad , \tag{V-24}$$

where

$$\Delta\beta_k = \text{tr}\,[P_{k+1}Q_k + P^*_{k+1}\hat{P}_{k/k}] + l^M_k(u^M_k) \quad . \qquad \text{(V-25)}$$

The optimum control law K_k and the cost matrices P_k and P^*_k are independent of $\hat{R}_k$ and H_k and, thus, are independent of the choice of u^M_k. *Therefore, the plant control policy can be determined separately from the measurement control policy.* Since the choice of u^M_k affects only $\hat{P}_{k/k}$ and l^M_k in Eq. (IV-23), the computation of the optimum u^M_k is *equivalent to the following nonlinear, deterministic control problem:* Minimize

$$J^* = \sum_{k=0}^{N} \{\text{tr}\,[P^*_{k+1}\hat{P}_{k/k}] + l^M_k(u^M_k)\} \quad , \qquad \text{(V-26)}$$

subject to Eq. (V-23) and the constraints given bv Eqs. (V-7) and (V-8). It is interesting to note that the matrix Riccati equation (V-23) plays the role of the state equation, with the elements of $\hat{P}_{k/k}$ corresponding to state variables. The results just presented can also be derived by use of Eqs. (V-16) through (V-19) in Eqs. (V-11) through (V-15); this derivation is presented in Appendix C.

Briefly, the procedure for solving this special case is as follows: Eq. (V-21) is solved to obtain the optimal control policy (*i.e.*, K_k) and the cost matrix P^*_k. Then the deterministic control problem described by Eqs. (V-7), (V-8), (V-23), and (V-26) is solved for the optimal sequence of measurement controls u^M_k. It should be emphasized that both K_k and u^M_k can be determined *a priori*.

These results can also be used in systems with suboptimal control and estimation of the form

$$u^P_k = -K'_k\hat{x}_{k/k}$$

$$\hat{x}_{k/k} = F_{k-1}\hat{x}_{k-1/k-1} + G_{k-1}u^P_{k-1} + \hat{K}'_k[z_k - H_k(F_{k-1}\hat{x}_{k-1/k-1} + G_{k-1}u^P_{k-1})] \quad . \qquad \text{(V-27)}$$

Since Eqs. (V-27) use the suboptimum gains K'_k and $\hat{K}'_k$, it is only necessary to use modified equations for $\Delta\beta_k$, P_k, P^*_k, and $\hat{P}_{k/k}$ which are given in Ref. 5 and will not be repeated here.

D. Examples

In order to demonstrate the principles developed in this section, two illustrative examples are presented.

1. Example I

For the first example a simple problem, in which the number of measurements is constrained, will be treated. Similar problems have been considered by Kushner.[8] Given:

The scalar plant

$$x_{k+1} = f_k x_k + u_k^P + w_k \quad ,$$

$$E(w_k) = 0 \quad , \quad E(w_k^2) = \hat{q}_k \quad . \tag{V-28}$$

The scalar measurement subsystem

$$z_k = x_k + v_k \quad ,$$

$$E(v_k) = 0 \quad , \quad E(v_k^2) = \hat{r}_k(u_k^M) \quad . \tag{V-29}$$

The constraint on the measurement control u_k^M is that M (where $M \leq N$) measurements must be made. If a measurement is made at time k, $\hat{r}_k = \gamma$; if no measurement is made at time k, $\hat{r}_k = \infty$.

The performance criterion is

$$J = E\left\{\sum_{k=0}^{N} (q_k x_k^2 + r_k u_k^{P^2}) + p_{N+1} x_{N+1}^2\right\} \quad . \tag{V-30}$$

It should be noted that the constraint on u_k^M given above is not a local constraint. Let $u_k^M = 1$ if a measurement is made and $u_k^M = 0$ if not, then define a new state variable y_k which obeys

$$y_0 = u_0^M$$

$$y_{k+1} = y_k + u_k^M \qquad k = 0, \ldots, N-1$$

$$y_N = M \quad . \tag{V-31}$$

The constraint implied by Eqs. (V-31) is the same as stated above.

As shown above, the determination of the optimal measurement policy reduces to the following nonlinear, deterministic control problem:

Minimize

$$J^* = \sum_{k=0}^{N} (q_k + f_k^2 p_{k+1} - p_k)\hat{p}_{k/k} \quad , \tag{V-32}$$

subject to*

$$\hat{p}_{k/k}^{-1} = (f_{k-1}^2 \hat{p}_{k-1/k-1} + \hat{q}_{k-1})^{-1} + \hat{r}_k^{-1}$$

$$k = 0, \ldots, N \quad , \tag{V-33}$$

where $\hat{p}_{k/k}$ is the conditional covariance of the error in the estimate of x_k, and p_k satisfies the equation

$$p_k = q_k + f_k^2 p_{k+1} - f_k^2 p_{k+1}^2 (p_{k+1} + r_k)^{-1}$$

$$k = N, \ldots, 0 \quad . \tag{V-34}$$

Consider this example with the following parameter values: $f_k = 0.9$, $\gamma = 1.0$, $q_k = 1.0$, $p_{N+1} = 1.0$, $r_k = 1.0$, $N = 3$, $M = 2$, $\hat{p}_{-1/-1} = 2.0$, for the two cases (a) zero disturbance noise, $\hat{q}_k = 0$; and (b) nonzero disturbance noise, $\hat{q}_k = 2$.

The results for Cases (a) and (b) are summarized in Figs. V-2 and V-3. The solid lines represent transitions from $k - 1$ to k when a measurement is made at time k; the dashed lines represent transitions from $k - 1$ to k when no measurement is made at time k. The values below the nodes at time k correspond to $\hat{p}_{k/k}$; the values above the nodes at time k correspond to the partial cost I_k^*, where

$$I_k^* \triangleq \sum_{i=0}^{k} (q_i + f_i^2 p_{i+1} - p_i)\hat{p}_{i/i} \quad . \tag{V-35}$$

It should be noted that certain transitions in the decision trees of Figs. V-2 and V-3 are not admissible, since two measurements must be made (*i.e.*, $M = 2$). The minimum value for J^* of Eq. (V-32) is shown circled in the figures.

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* For this case the covariance equation takes a simpler form if it is written in terms of the inverse.

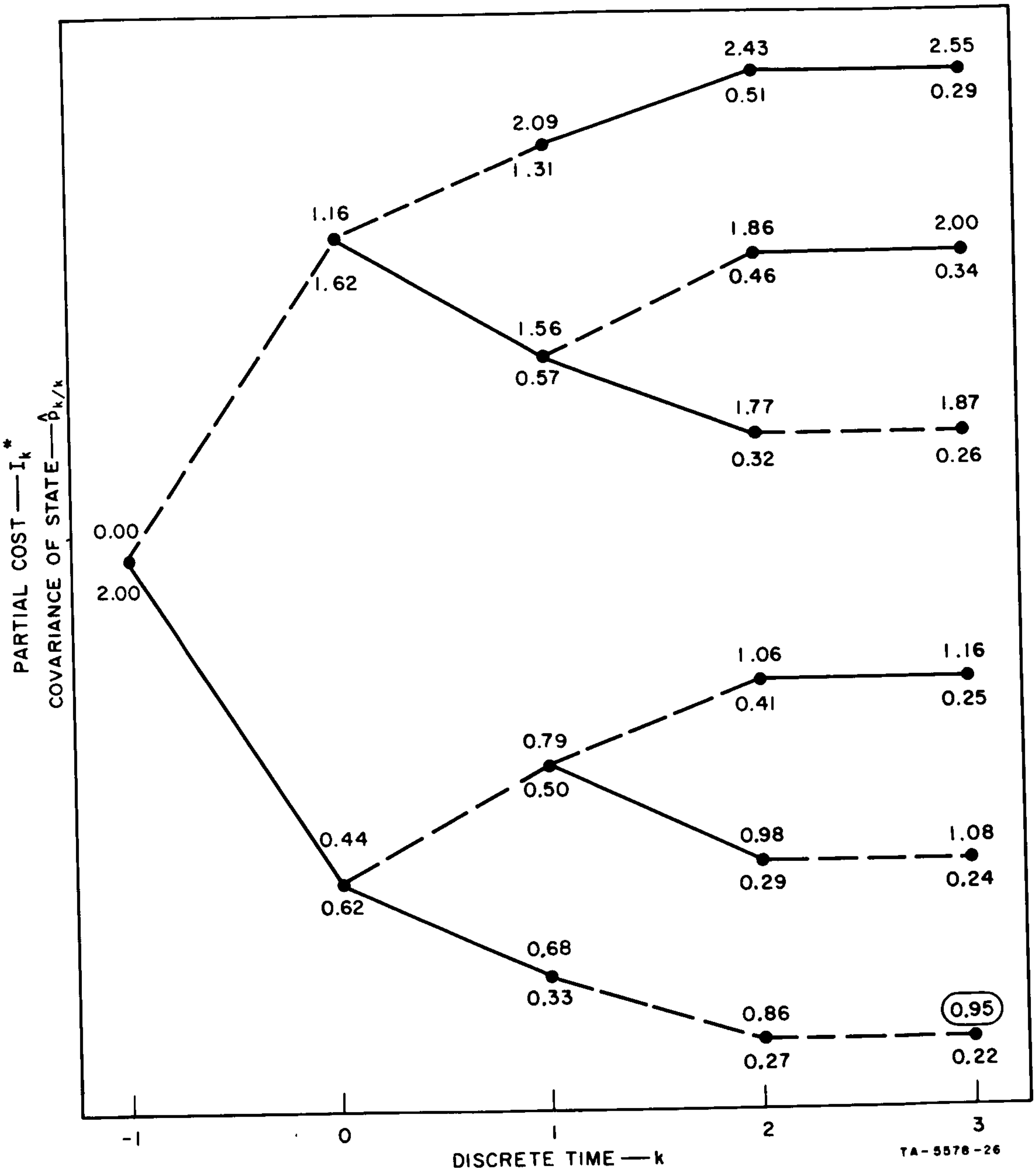

FIG. V-2 EXAMPLE I WITH $q_k = 0$

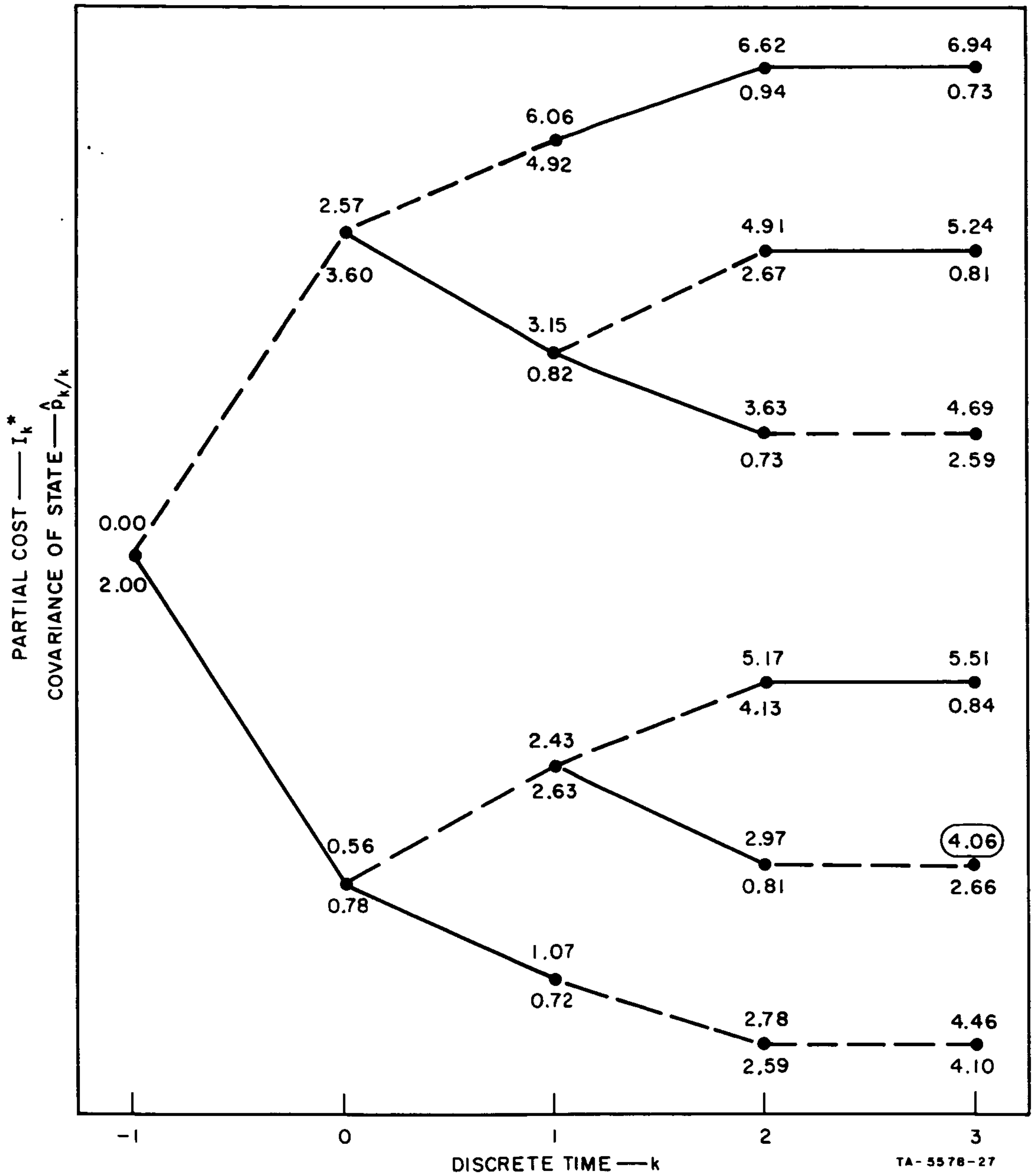

FIG. V-3 EXAMPLE I WITH $q_k = 2$

Hence, the optimum measurement policy is: Case (a) Make measurements at $k = 0, 1$. Case (b) Make measurements at $k = 0, 2$. These answers make sense from an intuitive point of view. With no disturbance the measurements should be made as soon as possible in order to remove the initial uncertainty. On the other hand, when there is a disturbance present, some measurements should be saved to determine the effect of the disturbance.

2. Example II

For a second example, consider the problem of terminal control using radar-derived measurements of the system state. The system under consideration, shown in Fig. V-4, is the discrete-time version of a double integrator:

$$x_{k+1} = F_k x_k + G_k u_k^P + w_k \quad , \tag{V-36}$$

where

$$x_k = \begin{bmatrix} x_k^{(1)} \\ \\ x_k^{(2)} \end{bmatrix} = \begin{bmatrix} \text{position} \\ \\ \text{velocity} \end{bmatrix} ,$$

$$F_k = \begin{bmatrix} 1 & 1 \\ \\ 0 & 1 \end{bmatrix} , \quad G_k = \begin{bmatrix} 1/2 \\ \\ 1 \end{bmatrix} ,$$

$$E(w_k) = 0 \quad , \quad E(w_k w_k^T) = \hat{Q} \quad .$$

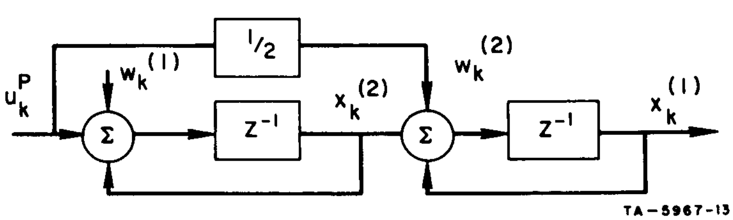

FIG. V-4 SYSTEM FOR EXAMPLE II

The radar measures position by the time delay of returning pulses and velocity by the Doppler shift of the pulses. For accurate position measurement a short pulse is desirable; whereas accurate velocity information necessitates a long pulse. Hence, the measurement subsystem, which measures both position and velocity, is governed by the observation equations:

$$z_k = x_k + v_k \quad ,$$

$$E(v_k) = 0 \quad , \quad E(v_k v_k^T) = \hat{R}_k = \begin{bmatrix} \left(\sigma_k^{(1)}\right)^2 & 0 \\ 0 & \left(\sigma_k^{(2)}\right)^2 \end{bmatrix} \quad . \tag{V-37}$$

The effect of the measurement control policy is to vary the operating mode of the radar and, hence, to change $\hat{R}_k$ in the appropriate manner. It will be assumed that $\sigma_k^{(1)}$ and $\sigma_k^{(2)}$, the standard deviations of the position and velocity measurements, are related by $\sigma_k^{(1)}\sigma_k^{(2)} = 1$. Since a terminal control problem is being considered, the performance index consists of a quadratic cost on the final position plus a quadratic cost on the plant control:

$$J = E\left\{ r \sum_{k=0}^{N} (u_k^P)^2 + \left(x_{N+1}^{(1)}\right)^2 \right\} \quad . \tag{V-38}$$

First consider the situation in which the measurement subsystem is constrained to operate in one of two modes at each time instant: in the 0 - mode, velocity is measured relatively well and position relatively poorly; in the 1 - mode, vice versa. For purposes of exposition, it is convenient to let the corresponding measurement noise covariance matrices take the form

$$\hat{R}_k^0 = \begin{bmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{bmatrix} \quad , \quad \hat{R}_k^1 = \begin{bmatrix} 1/8\alpha & 0 \\ 0 & 8\alpha \end{bmatrix} \quad . \tag{V-39}$$

It is desired to find the optimal sequence of $\hat{R}_k^0$ and $\hat{R}_k^1$, where α is a parameter.

For several values of α the optimum measurement control sequences, as well as the optimal performances, were computed using forward dynamic programming. The results are displayed in Fig. V-5, which also lists the values of the other system parameters used and gives the performances with perfect measurements and using either all $\hat{R}_k^0$ or all $\hat{R}_k^1$. It is interesting to note that, although the 1 - mode is the best mode to use if only one mode is allowed, in the two-mode situation the 0 - mode is used more often than the 1 - mode.

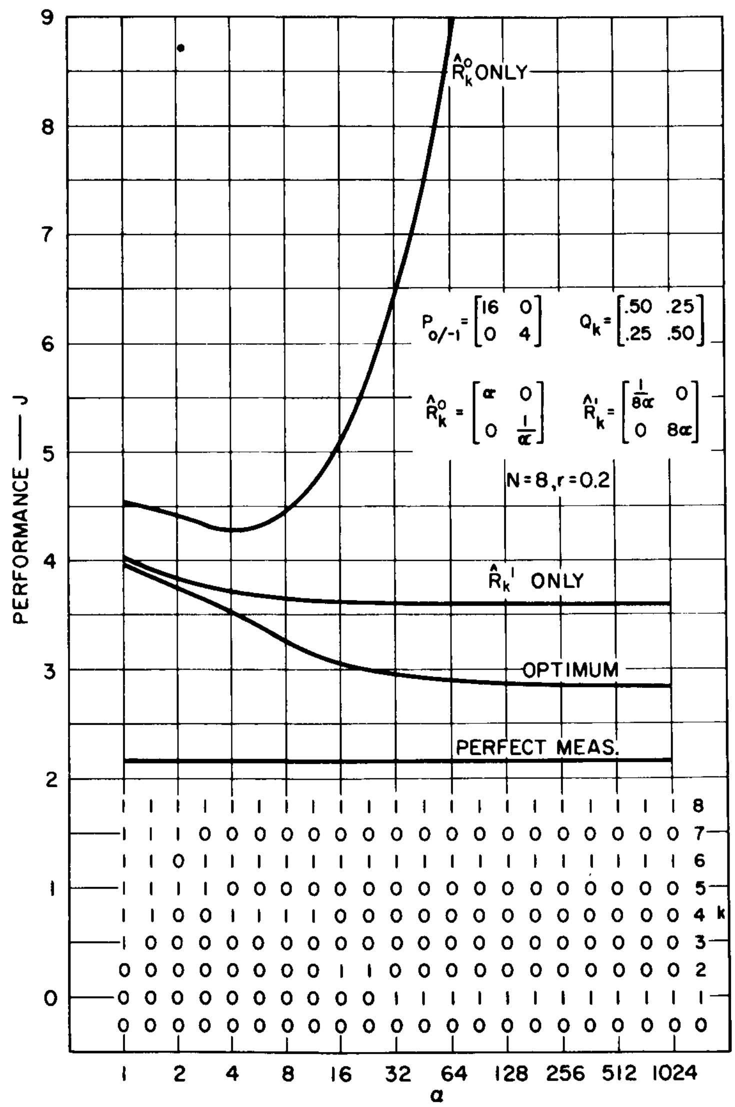

FIG. V-5 RESULTS FOR EXAMPLE II

Now consider the case of a continuous mode measurement subsystem, which is described by

$$\hat{R}_k = \begin{bmatrix} u_k^M & 0 \\ 0 & 1/u_k^M \end{bmatrix} , \qquad \text{(V-40)}$$

where

$$0.000488 \leq u_k^M \leq 256 \quad .$$

Table V-1 gives the sequence of u_k^M computed using the gradient method, which was initialized with the optimum "bang-bang" solution (*i.e.*, the solution when u_k^M is restricted to its extreme values). Note that the only difference between the optimum "bang-bang" solution and the continuous mode solution occurs at $k = 0$; furthermore, the improvement in performance is only from $J = 2.8658$ to 2.8657.

Unfortunately, as is characteristic of gradient methods, one can only be sure that the solution displayed in Table V-1 is a relative optimum. Indeed using other initial sequences the gradient method converged to different relative optima; none of which, however, had a performance as good as that for the sequence given in Table V-1. The fact that the optimum "bang-bang" sequence was used to initialize the computations lends some credence to the belief that the result obtained is the absolute optimum.

Table V-1

OPTIMUM MEASUREMENT CONTROLS

k	u_k^M
0	82.8
1	0.000488
2	256
3	256
4	256
5	256
6	0.000488
7	256
8	0.000488

E. Conclusions

In this section the concept of measurement adaptive systems was formulated and solved in the general case as well as in the special case of linear systems with Gaussian perturbations and quadratic cost. In this special case, the resulting problem of solving for the optimum measurement control reduces to one of classical optimal control, where the elements of the covariance matrix of the state estimate act as state variables and the matrix Riccati equation plays the role of the equations of motion.

As pointed out earlier, the problem of finding the optimal measurement control policy for a linear system with Gaussian disturbances and quadratic costs reduces to a nonlinear, deterministic control problem. Two basic computational procedures exist for the solution of such a problem: dynamic programming and the gradient method. Dynamic programming, because it is a global optimization procedure, will give the absolute optimum if it is computationally feasible; but because it is global, it suffers from the curse of dimensionality. The gradient method, because it is a local optimization procedure, is likely to be computationally feasible for more complicated problems than dynamic programming, but can be only guaranteed to give a relative optimum. The above comments apply, of course, to any deterministic control problem; what makes the problem of finding measurement controls nontrivial, if the second example is any indication, is the prevalence of relative optima. This example also illustrates a possible escape from this dilemma: Use dynamic programming on a simplified version of the problem and then use the gradient procedure on the complete problem to refine the control sequence obtained by dynamic programming.

VI APPLICATIONS

A. Introduction

The research efforts carried out under Contracts NAS 2-2457 and NAS 2-3476 have been predominantly theoretical in nature and no attempt has been made to apply the results obtained to a specified practical problem. However, in order to ensure that the research carried out would eventually respond to practical needs, a minor effort was devoted to the study of aircraft avionic systems, air traffic control procedures, and V/STOL aircraft control requirements. Rather than describe in detail the functioning of present and projected systems and procedures and to suggest how the techniques developed might be used to improve specific items, we prefer to discuss in a general way the applicability of these techniques to the problems of system design and operation. It is stressed however, that the detailed study of such practical applications as aircraft avionics systems has suggested the general conclusions to be reported in this section and has strongly influenced the design methodology for control and guidance systems required to operate in the presence of uncertainty, as discussed in Secs. II and III.

Looking more closely at the theory developed, one may divide it into two parts as follows:

(1) *The analysis of the effects of uncertainty by applying sensitivity theory to the cost function and the equations of motion*—This part allows one to assess in the simplest possible manner the degrading effects of plant and sensor noise and suggests how the system should be "adapted" to parameter uncertainties.

(2) *The synthesis of systems where the degrading effects of uncertainty are minimized*—For this part, several complementary approaches of increasing complexity can be taken, *i.e.*,

   (a) Go from an open-loop to a closed-loop system

   (b) Attempt to estimate imperfectly known parameters and adapt the law of control accordingly

(c) Include the stochastic and dual effects, which tend to shift the nominal trajectory into a region of space where the intensity and/or the degrading effect of plant and sensor noise is reduced.

We also wish to stress that, although primary concern was given to guidance systems and control systems operating on physical plants, the theories evolved in the course of the study apply equally to problems of system planning and operation involving nonphysical plants, such as management processes, planning of experiments, establishment of system evaluation procedures, design of systems containing humans in a decision function, allocation of *R* and *D* funds, etc. The reason for this general applicability is that the mathematical models for the equations of motion (state transition equations), the cost function and the uncertainties involving the model, the forces affecting the transition equations and the measurement of state are very similar, if not identical, for all the problem areas mentioned above.

B. Design of Guidance and Control Systems

Under this heading, we include the real-time systems where a given physical plant, *e.g.*, aircraft, space vehicle, or tracking antenna is to be controlled optimally. In applications of this kind, it is common to have several separate control systems, the functions of which are entirely uncoupled or loosely coupled. For example, in an aircraft application, we may find separate systems for automatic navigation, attitude/altitude/heading control (autopilot), cruise control, and automatic landing. These systems are usually treated separately because the objectives (cost functions) are quite different. The main objective of precise navigation is collision avoidance; the benefit derived from the autopilot is ease of control and smooth flight; the purpose of cruise control is fuel economy and concommitant maximization of payload; automatic landing, finally, allows the airplane to reach its destination under all weather conditions.

In situations such as these, the techniques developed are applied independently to each of the systems discussed. The most convenient and logical step-by-step procedure to apply the techniques was discussed in greater detail in Sec. II and is briefly summarized here as follows:

(1) Optimize the control system, assuming a perfectly known model and no plant or measurement noise, by means of the well-known deterministic optimization procedures.

(2) Investigate the sensitivity of the cost function with respect to these three classes of uncertainty using (in general) second-order sensitivity theory. This sensitivity analysis determines approximately the information requirements, that is, the accuracy of the model, the magnitude of the perturbations that are tolerable, and the state-variables that must be estimated, together with the accuracy required.

(3) Estimate the utilization of this information by going from an open-loop configuration to a closed-loop configuration; if necessary, supplement the closed-loop configuration by adaptation. Finally, for certain guidance and control problems, compensate for the stochastic and dual effects by shifting the nominal trajectory.

It is clear that the analysis part of paragraph 2 and the synthesis part of (3) are often intermixed. For example, to analyze which state variables need to be measured, a closed-loop configuration must be assumed; the closed-loop configuration is the first step of the synthesis procedure.

C. Reliability Considerations

The design and evaluation of a system from the point of view of component and subsystem reliability can be performed in much the same way. First, one obtains the performance, assuming that all the components are functioning properly. Next, one allows the components to fail and calculates the performance of the partially failed system. These failures can be viewed as large parameter changes in the differential equations describing the system. If one associates probabilities with these component failures, then the statistics of the cost or performance function can be derived. This completes the analysis half of the procedure.

The aim of the synthesis part consists of minimizing the performance degradation caused by component malfunctions. This can be accomplished during the system design stage by use of more reliable components or by component duplication. The sensitivity of the expected performance with respect to the various component failures identifies those components that should be made more reliable. From this approach, it is apparent that requiring uniform component reliability is not in general a good practice; for some components, a reliability of 0.9 may be adequate, whereas for others 0.999 may not be sufficient.

Instead of increasing the reliability of the key components in the design state, one may also minimize the degrading effect of a component failure during the operating stage by appropriate control action. After a component failure has been detected, the controller maximizes performance subject to the equations of motion of the partially failed system. This approach can be viewed as a form of adaptive control. As a simple example of this, one may consider a linear control system with quadratic cost function where the state is observed by a set of noisy sensors $S_1, \ldots, S_n$. The estimator gains are determined by the covariances of these sensors. Now, assume that one sensor S_i fails, meaning that its covariance becomes infinite.. If this fact is detected, the estimator gains are changed and the estimation process remains optimal for the partially failed instrumentation system. If the failure is not detected, the estimator output may contain unacceptable errors. On the other hand, the system performance obtained with the adapted estimator may be unacceptable, in which case the sensor S_i must be duplicated or additional variables must be measured.

D. Design of Experiments

The purpose of designing an experiment is to acquire information in order to improve performance. Wind-tunnel tests, for example, provide aerodynamic parameters, knowledge of which is necessary to design, among others, an efficient autopilot. In other words, knowledge of these parameters is important only to the extent that it influences the autopilot design, and not *per se*. The ultimate aim, namely system performance, fixes the information state that must exist at the end of the experiment.

The problem then is to get from the present information state (initial uncertainty about the aerodynamic parameters) to the required information state by a suitable selection of experiments; the nature and sequence of the tests may be viewed as the control variables. Since experiments are costly, it is reasonable to select, among the many feasible sequences of tests, that which provides the desired information state with least cost. This is a variational optimization problem in which the state variables are the conditional probability density functions characterizing the information state and not the customary energy-storage output or resource variables.

The optimum allocation of research and development funds represents a very similar class of problems. The purpose of initiating an *R* and *D* project is to improve the information state with the ultimate aim of achieving desired system performance. Again, this ultimate aim determines the required information state which it is then desired to reach with the least expenditure of funds, or in least time, etc.

The analytical foundation of the applications discussed in this paragraph are given in Sec. IV entitled "Optimum Control of Measurement Subsystems."

E. Systems Containing Man in a Decision Function

The techniques discussed in this report were primarily aimed at fully automatic dynamic systems. In principle, the key ideas can be applied to systems containing humans responsible for making decisions. We do not consider in this paragraph situations where the "transfer function" of man is of concern, for example, manual tracking applications; rather, we concern ourselves with situations where the time scale is such that a rational decision can be made based on the information made available to the human decision-maker. The duties of an air traffic controller might fall into this category.

One of the difficult design problems arising in systems of this kind is the selection of the information that must be made available and the manner in which it should be displayed. The techniques discussed in this report can be applied to this problem as follows:

(1) Determine the system performance assuming perfect information and perfect decisions.

(2) Determine the expected system degradation for imperfect or incomplete information, but perfect decisions. This determines the required information sources and displays assuming a perfectly trained and intelligent crew.

(3) Determine the expected performances with an actual crew, the decisions of which are not always perfect. Compensate for this imperfection by providing more information or by using automatic data-processing aids to improve the quality of the decisions.

F. System Planning Problems

Planning the evolution in time of a system or facility can be accomplished using the techniques described in this report. To illustrate what we mean by a facility required to evolve with time, we may think of the deep-space communication network operated by the NASA. The performance (*e.g.*, bandwidth) of this facility is required to improve with time in accordance with the mission planned.

The problem of the planner is to add equipment (*e.g.*, antennas) of characteristics to be determined at suitable time intervals in the future; these added components become the state variables of the problem. The performance function is a measure of how well these added components satisfy the demands placed upon them over the planning interval, which may be of the order of 20 years.

The first step in the procedure consists of finding the characteristics and implementation schedule, assuming

(1) Perfect knowledge of the demands placed upon the system, *e.g.*, missions taking place during the planning interval.

(2) Perfect knowledge of the technical and economic characteristics of forthcoming equipment generations. In the second step, the degrading effect upon performance of uncertainty (demand and equipment characteristics) is analyzed. It must be kept in mind here that the planning process is characterized by a decision rule (*i.e.*, law of control) because the planning decisions are reviewed repeatedly to account for the information acquired in the meanwhile.

(3) During the third step, the decisions are modified to minimize the degrading effects of uncertainty. It seems that the most common protection against these degrading effects is to provide for enough intermediate decision options. In the context of our planning example, this might mean that the purchase of communication facilities believed to be necessary for future missions should be delayed, even at the cost of more expensive procurement, until the exact nature of the missions and the characteristics of forthcoming equipment generations is firmly established.

APPENDIX A

SENSITIVITY FOR SYSTEMS WITH STATIC COST FUNCTION AND ALGEBRAIC EQUALITY CONSTRAINTS

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APPENDIX A

SENSITIVITY FOR SYSTEMS WITH STATIC COST FUNCTION AND ALGEBRAIC EQUALITY CONSTRAINTS

1. Problem Formulation

Given: the scalar cost function

$$J = F(x,u) \tag{A-1}$$

of the dependent and independent (control) variables x and u, both of which are vectors of dimension n and m, respectively, and the n equality constraints

$$g(x,u,p) = 0 \tag{A-2}$$

where p is a parameter vector,

Find:

(1) The sensitivity relations between the problem variables x, u and p

(2) The first-order sensitivity of the cost function F with respect to x, u, and p

(3) The second-order sensitivity of the cost function F with respect to x, u, and p.

In the interest of keeping the results simple, it is assumed that the variables x and u are not subjected to inequality constraints of the form

$$x_i \le X_i \quad ; \quad u_j \le U_j \quad .$$

If inequality constraints of this type have a dominant effect upon the problem solution, they may be taken into account by adding appropriate penalty terms to the cost function (A-1). It is furthermore assumed that the functions F and g are continuous and twice differentiable with respect to their arguments.

2. Sensitivity Relations Between the Problem Variables x, u, and p

It is assumed that a solution x^*, corresponding to the nominal control vector u^* and the nominal parameter vector p^*, has already been obtained:

$$g(x^*, u^*, p^*) = 0 \quad . \tag{A-3}$$

It is now desired to determine the changes Δx which would result from small changes Δu and Δp away from the nominals u^* and p^*. A Taylor-series expansion of Eq. (A-3) provides this relation:

$$g(x^* + \Delta x, u^* + \Delta u, p^* + \Delta p) = 0 \cong g(x^*, u^*, p^*) + g_x \Delta x + g_u \Delta u + g_p \Delta p \quad . \tag{A-4}$$

In view of Eq. (A-4), it follows that

$$g_x \Delta x + g_u \Delta u + g_p \Delta p = 0 \tag{A-5}$$

$$\Delta x = -g_x^{-1} g_u \Delta u - g_x^{-1} g_p \Delta p \quad , \tag{A-6}$$

where the Jacobian matrices g_x, g_u, g_p are evaluated at the nominal solution x^*, u^*, p^*. For convenience of notation, Eq. (A-6) is rewritten in terms of the *sensitivity matrices* S_u and S_p as

$$\Delta x = S_u \Delta u + S_p \Delta p \quad , \tag{A-7}$$

with

$$S_u = -g_x^{-1} g_u \tag{A-8}$$

$$S_p = -g_x^{-1} g_p \quad . \tag{A-9}$$

Example:

For the aircraft cruise control problem also discussed in Sec. II, the cost function F is given by

$$F = \frac{\sigma T}{cM} \quad , \tag{A-10}$$

σ = specific fuel consumption = 0.29 10^{-3} lb $\sec^{-1}$/lb of thrust

T = thrust, in lb, here the dependent variable x_1

c = speed of sound = 968.1 ft/sec at the prescribed altitude of 50,000 ft

M = Mach number, here the independent problem variable u.

The equality constraints are:

$$kC_{L_\alpha}\alpha \frac{\rho c^2 M^2}{2} S - mg + T \sin(\alpha + \epsilon) = 0 \qquad \text{(A-11)}$$

$$(C_{D_0} + k^2 \eta C_{L_\alpha}\alpha^2)\rho \frac{c^2 M^2}{2} S - T \cos(\alpha + \epsilon) = 0 \quad , \qquad \text{(A-12)}$$

where

α = angle of attack, here the dependent variable x_2

ϵ = fixed angle of 0.05 rad given by aircraft geometry

mg = weight of aircraft, nominally 34,000 lb; here, mg is a variable parameter p_1

S = wing surface, nominally 530 ft^2; here S is an uncertain parameter p_2

C_{L_α}, C_{D_0} and η = aerodynamic coefficients the variation with M of which is given in Fig. II-1

k = constant = $180/\pi$ = 57.2947795.

At the operating point defined by

M = 2.683

T = 6133 lb

α = $5.716\ 10^{-4}$ rad

F = $6.8474\ 10^{-4}$ lb/ft $\cong$ 3.423 lb/mile ,

the sensitivity matrices S_u and S_p are

$$S_u = \begin{bmatrix} 2.859\ 10^{3} \\ -2.377\ 10^{-4} \end{bmatrix} \qquad S_p = \begin{bmatrix} 6.536\ 10^{-2} & 7.378 \\ 1.690\ 10^{-8} & -1.084\ 10^{-6} \end{bmatrix} .$$

The interpretation of these matrices is as follows:

For S_u; a 1-percent change in Mach number ($\Delta M = 0.02683$) requires a change of thrust ΔT of $2859 \cdot 0.02683 = 77.2$ lb.

For S_p; a 1-percent change in weight ($\Delta mg = 340$) requires a change of thrust ΔT of $0.0653 \cdot 340 = 22.2$ lb.

3. First-Order Sensitivity of the Cost Function F with Respect to x, u, p

For sufficiently small variations Δu and Δp, a Taylor-series expansion limited to the first term of Eqs. (A-1) and (A-2) yields for the nominal x, u, and p:

$$\Delta F = F_x \Delta x + F_u \Delta u \tag{A-13}$$

$$g_x \Delta x + g_u \Delta u + g_p \Delta p = 0 \quad , \tag{A-14}$$

where the row vectors F_x and F_u and the rectangular matrices g_x, g_u, and g_p are the partial derivatives $\partial F/\partial x_j$, $\partial F/\partial u_j$, $\partial g_i/\partial x_j$, $\partial g_i/\partial u_j$, and $\partial g_i/\partial p_j$ evaluated at x, u, and p. Elimination of the dependent variation Δx from Eqs. (A-13) and (A-14) yields

$$\Delta F = (F_u - F_x g_x^{-1} g_u)\Delta u - F_x g_x^{-1} g_p \Delta_p \quad . \tag{A-15}$$

In the interest of more compact notation, the Lagrangian $\mathcal{L}(x,u,p)$ will be defined as

$$\mathcal{L} = F(x,u) + \lambda^T g(x,u,p) \quad , \tag{A-16}$$

where the dual vector λ of dimension n is defined by

$$F_x(x,u) + \lambda^T g_x(x,u,p) = 0 \quad . \tag{A-17}$$

With the two definitions of Eqs. (A-16) and (A-17), one can prove immediately that an alternative expression of Eq. (A-13) is

$$\Delta F = \mathcal{L}_u \Delta u + \mathcal{L}_p \Delta p \quad , \tag{A-18}$$

where $\mathcal{L}_u$ and $\mathcal{L}_p$ are again row vectors.

If the control u selected for the nominal is an optimum control (*i.e.*, minimizes the cost F), the sensitivity of ΔF with respect to Δu must clearly be zero. This is an alternative way of stating the well-known necessary condition for an optimum:

$$\mathcal{L}_u = 0 \quad .$$

Example:

For the aircraft cruise control system under discussion, the Lagrangian function $\mathcal{L}$ is

$$\mathcal{L} = \frac{\sigma T}{cM} + \lambda_1 \left[k C_{L_\alpha} \alpha \frac{\rho c^2 M^2}{2} S - mg + T \sin (\alpha + \epsilon) \right]$$

$$+ \lambda_2 \left[(C_{D_0} + k^2 \eta C_{L_\alpha} \alpha^2) \rho \frac{c^2 M^2}{2} S - T \cos (\alpha + \epsilon) \right] \quad . \qquad \text{(A-19)}$$

At the operating point defined by $M = 2.683 \ldots$, the partial $\mathcal{L}_u$ is zero indicating that the chosen value of M minimizes the cost function F. The partial $\mathcal{L}_{p_1}$ corresponding to this operating point is

$$\mathcal{L}_{p_1} = -\lambda_1 = 7.29 \; 10^{-9} \quad .$$

This means that a 1-percent increase of weight ($\Delta mg = 340$ lb) causes an increase in F of $2.48 \; 10^{-6}$ lb/ft, equivalent to a percent increase $100 \; \Delta F/F$ of 0.36 percent.

4. Second-Order Sensitivity of the Cost Function F with Respect to x, u, and p

A Taylor-series expansion limited to the second term of Eqs. (A-1) and (A-2) yields

$$\Delta F = F_x \Delta x + F_u \Delta u + \frac{1}{2} \Delta x^T F_{xx} \Delta x + \frac{1}{2} \Delta u^T F_{uu} \Delta u + \Delta x^T F_{xu} \Delta u \quad . \qquad \text{(A-20)}$$

For the ith equality constraint, a similar Taylor-series expansion yields

$$g_x^i \Delta x + g_u^i \Delta u + g_p^i \Delta p + \frac{1}{2} \Delta x^T g_{xx}^i \Delta x + \frac{1}{2} \Delta u^T g_{uu}^i \Delta u$$

$$+ \frac{1}{2} \Delta p^T g_{pp}^i \Delta p + \Delta x^T g_{xu}^i \Delta u + \Delta u^T g_{up}^i \Delta p + \Delta p^T g_{px}^i \Delta x = 0 \quad . \qquad \text{(A-21)}$$

With the definition of the Lagrangian $\mathcal{L}$ in Eq. (A-16), multiplication of each equation of (A-21) by λ_i and addition of these equations to (A-20) yields

$$\Delta F = \mathcal{L}_u \Delta u + \mathcal{L}_p \Delta p + \frac{1}{2} \Delta x^T \mathcal{L}_{xx} \Delta x + \frac{1}{2} \Delta u^T \mathcal{L}_{uu} \Delta u$$

$$+ \frac{1}{2} \Delta p^T \mathcal{L}_{pp} \Delta p + \Delta x^T \mathcal{L}_{xu} \Delta u + \Delta u^T \mathcal{L}_{up} \Delta p + \Delta p^T \mathcal{L}_{px} \Delta x \tag{A-22}$$

The dependent variation Δx is expressed in terms of the independent variations Δu and Δp through the sensitivity equations

$$\Delta x = S_u \Delta u + S_p \Delta p \quad . \tag{A-7}$$

Substitution of this equation into Eq. (A-22) yields

$$\Delta F = \mathcal{L}_u \Delta u + \mathcal{L}_p \Delta p + \frac{1}{2} \Delta u^T (\mathcal{L}_{uu} + S_u^T \mathcal{L}_{xx} S_u + 2 S_u^T \mathcal{L}_{xu}) \Delta u$$

$$+ \frac{1}{2} \Delta p^T (\mathcal{L}_{pp} + S_p^T \mathcal{L}_{xx} S_p + 2 \mathcal{L}_{px} S_p) \Delta p$$

$$+ \Delta u^T (\mathcal{L}_{up} + S_u^T \mathcal{L}_{xx} S_p + \mathcal{L}_{xu} S_p + S_u^T \mathcal{L}_{px}^T) \Delta p \quad .$$

For more compact notation, this equation is written as

$$\Delta F = \mathcal{L}_u \Delta u + \mathcal{L}_p \Delta p + \frac{1}{2} \Delta u^T \mathcal{L}^*_{uu} \Delta u + \frac{1}{2} \Delta p \mathcal{L}^*_{pp} \Delta p + \Delta u^T \mathcal{L}^*_{up} \Delta p \quad , \tag{A-23}$$

where the matrices $\mathcal{L}^*_{uu}$, $\mathcal{L}^*_{pp}$, and $\mathcal{L}^*_{up}$ are given by

$$\mathcal{L}^*_{uu} = \mathcal{L}_{uu} + S_u^T \mathcal{L}_{xx} S_u + 2 S_u^T \mathcal{L}_{xu}$$

$$\mathcal{L}^*_{pp} = \mathcal{L}_{pp} + S_p^T \mathcal{L}_{xx} S_p + 2 \mathcal{L}_{px} S_p$$

$$\mathcal{L}^*_{up} = \mathcal{L}_{up} + S_u^T \mathcal{L}_{xx} S_p + \mathcal{L}_{xu}^T S_p + S_u^T \mathcal{L}_{px}^T \quad .$$

Example:

For the cruise control problem under discussion, the matrices $\mathcal{L}^*_{uu}$, $\mathcal{L}^*_{up}$ and $\mathcal{L}^*_{pp}$ are given below for the operating point defined by $M = 2.683$ ...

$$\mathcal{L}^*_{uu} = 2.199\ 10^{-4}$$

$$\mathcal{L}^*_{up} = [-5.745\ 10^{-9} \qquad 3.685\ 10^{-7}]$$

$$\mathcal{L}^*_{pp} = \begin{bmatrix} 2.151 \cdot 10^{-13} & -1.380 \cdot 10^{-11} \\ -1.380 \cdot 10^{-11} & 8.854 \cdot 10^{-10} \end{bmatrix} .$$

Using the perturbation model of Eq. (A-23) for a variation Δmg of 3400 lb (10 percent), we calculate a variation of cost

$$\Delta F = 26.14\ 10^{-6} .$$

The accuracy of this model was checked by directly computing the cost F for the new weight of 37400 lb. The true increase in cost was found to be

$$\Delta F = 26.22 \cdot 10^{-6} .$$

5. Applications of the Second-Order Perturbation Model

The perturbation model of Eq. (A-23) can be used for numerous purposes discussed elsewhere in the report—see notably Sec. II. The following two applications are singled out here in view of their great practical usefulness.

6. Optimization by Small Changes Δu of Control

Expression (A-23) may be used to advantage to optimize a real-time control system with little computational effort, once the optimal solution u° corresponding to the nominal parameter value p° has been obtained. Instead of recomputing a new optimal control with the original optimization equations every time a parameter change Δp occurs, it is usually much more efficient to compute the optimum variations Δu by minimizing Eq. (A-23). The minimizing value of Δu can be explicitly written as

$$\Delta u = -\mathcal{L}^{*-1}_{uu}\mathcal{L}^*_{up}\Delta p = K\Delta p \quad , \qquad \text{(A-24)}$$

where the matrix K is the optimum law of control for small variations. Consequently, as long as the parameter variations remain sufficiently small, optimization can be performed simply by adding the correction Δu of Eq. (A-24) to the nominal u°.

Example:

For the cruise control example under discussion, the control matrix K is

$$K = [2.611 \cdot 10^{-5} \quad -1.675 \cdot 10^{-3}] \quad .$$

7. Waste Caused by Inaccurate Information on p

Equations (A-23) and (A-24) may also be used to assess the unnecessary cost (or waste) caused by inaccurate information on p. This waste, if expressed in terms of the inaccuracy Δp, indicates the relative importance of measuring and telemetering the parameter p accurately.

The practical situation of concern in this section is shown in Fig. A-1. This system is subjected to the input signal $u + \Delta u$ put out by the controller. This signal would be u if the parameter value fed to the controller were p; since in actuality the controller receives $p + \Delta p$, its output will be $u + \Delta u$. The relation between u and p is the "law of control"

$$u = \varphi(p) \tag{A-25}$$

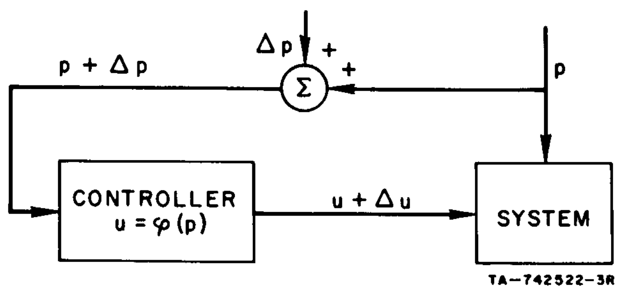

FIG. A-1 STATIC CONTROL SYSTEM RELATING THE CONTROL u TO THE PARAMETER p (Δp is a measurement inaccuracy causing control deviation Δu) implemented in the controller. For small changes Δp, the resulting changes Δu may be approximated as

$$\Delta u = \varphi_p \Delta p \quad , \tag{A-26}$$

if the function φ is sufficiently smooth about p.

Equation (A-23) next relates the approximate cost variation ΔF to the control deviation Δu by

$$\Delta F = \mathfrak{L}_u \Delta u + \frac{1}{2} \Delta u^T \mathfrak{L}^*_{uu} \Delta u \quad . \tag{A-27}$$

The variations Δp are omitted from Eq. (A-27), since the system receives p and not $p + \Delta p$.

If the law of control of Eq. (A-24) is optimum in the sense that u minimizes F for p, then the term $\mathfrak{L}_u$ is zero and the cost variation is

$$\Delta F = \frac{1}{2} \Delta u^T \mathfrak{L}^*_{uu} \Delta u \quad , \tag{A-28}$$

where Δu is related to Δp by Eq. (A-24). The combination of Eqs. (A-28) and (A-24) yields

$$\Delta F = \Delta p^T \Omega \Delta p \quad , \tag{A-29}$$

where the matrix Ω is given by

$$\Omega = \frac{1}{2} K^T \mathfrak{L}^*_{uu} K \quad . \tag{A-30}$$

If the inaccuracy Δp is a random time-uncorrelated measurement noise of zero mean

$$\hat{Q} \triangleq E\{\Delta p \Delta p^T\} \quad , \tag{A-31}$$

where the symbol E denotes expectation, then the average cost increase $E\{\Delta F\}$ caused by the random process Δp is

$$E\{\Delta F\} = E\left\{\sum_i \sum_j \Omega_{ij} \Delta p_i \Delta p_j\right\} = \sum_i \sum_j \Omega_{ij} \hat{Q}_{ij} \quad . \tag{A-32}$$

This equation is written compactly as

$$E\{\Delta F\} = \mathrm{tr}\,[\Omega\hat{Q}] \quad , \tag{A-33}$$

where the symbol "tr" denotes the trace of the matrix $\Omega\hat{Q}$.

Equations (A-29) and (A-33) readily point out the deleterious effects of bad measurements upon cost. Hence, they allow the system designer to allocate high-quality instruments and telemetry links to the sensitive measurements and vice versa.

Example:

For the aircraft cruise control system under discussion, the matrix Ω is

$$\Omega = \begin{bmatrix} 7.502 \cdot 10^{-14} & -4.813 \cdot 10^{-12} \\ -4.813 \cdot 10^{-12} & 3.088 \cdot 10^{-10} \end{bmatrix} .$$

This means that a 2-percent measurement error in the actual weight ($\Delta mg = 680$ lb) will lead to an *unnecessary* increase in cost of

$$\Delta F = 3.44 \cdot 10^{-8} \text{ lb/ft}$$

or in percent

$$100 \frac{\Delta F}{F} = 0.009 \text{ percent.}$$

APPENDIX B

DERIVATION OF THE EQUATIONS FOR SECTION III

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APPENDIX B

DERIVATION OF THE EQUATIONS FOR SECTION III

1. Computation of $E[(x - \hat{x})/Z]$ and $\hat{P}$

In this portion, approximate equations for $E\{x - \hat{x}/Z\}$ and $\hat{P}$ will be derived. From Eqs. (III-1), (III-2) and (III-6)

$$
\begin{aligned}
x(t + \Delta t) - \hat{x}(t + \Delta t) &= x(t) - \hat{x}(t) + \int_0^{\Delta t} [\dot{x}(t + \tau) - \dot{\hat{x}}(t + \tau)]d\tau \\
&\approx \{f[x\ t] - f[\hat{x}\ t] - \frac{1}{2} f^{o}_{xx} o\hat{P} + \hat{K}H[x(t) - \hat{x}(t)]\}\Delta t \\
&\quad + \int_0^{\Delta t} w(t + \tau)d\tau + \hat{K}\int_T^{\Delta} v(t + \tau)d\tau + o(\Delta t) \quad .
\end{aligned}
\tag{B-1}
$$

From Taylor's series expansion

$$f(x) - f(\hat{x}) \approx f^{o}_{x}(x - \hat{x}) + \frac{1}{2} f^{o}_{xx} o[(x - \hat{x})(x - \hat{x})^T] \quad . \tag{B-2}$$

However, $E(w/Z)$ and $E(v/Z)$ are zero; hence

$$
\begin{aligned}
E[\dot{x}(t) - \dot{\hat{x}}(t)/Z(t)] &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} E[x(t + \Delta t) - \hat{x}(t + \Delta t) - x(t) + \hat{x}(t)/Z(t)] \\
&\approx (f^{o}_{x} - \hat{K}H)E[x(t) - \hat{x}(t)/Z(t)] \quad .
\end{aligned}
\tag{B-3}
$$

$E[x(0)] = \hat{x}(0)$ and, therefore, $E[x(t) - \hat{x}(t)/Z(t)]$ is zero. Since

$$E[\int_0^{\Delta t} w(t + \tau)dt \cdot \int_0^{\Delta t} w(t + \tau)d\tau/Z(t)] = \hat{Q}(t)\Delta t + o(\Delta t) \quad , \tag{B-4}$$

and

$$E[\int_0^{\Delta t} v(t + \tau)d\tau \cdot \int_0^{\Delta t} v(t + \tau)d\tau/Z(t)] = \hat{R}(t)\Delta t + o(\Delta t) \quad , \tag{B-5}$$

it follows that

$$\dot{\hat{P}} \triangleq \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Big(E\{[x(t + \Delta t) - \hat{x}(t + \Delta t)][x(t + \Delta t) - \hat{x}(t + \Delta t)]^T$$

$$- [x(t) + \hat{x}(t)][x(t) - \hat{x}(t)]^T / Z(t)\} \Big)$$

$$\approx (f_x^o - \hat{K}H)\hat{P} + \hat{P}(f_x^o - \hat{K}H)^T + \hat{Q} + \hat{K}\hat{R}\hat{K}^T \quad . \tag{B-6}$$

2. <u>Computation of J</u>

a. <u>Statement of the H-J Equation</u>

To describe the control system it is convenient to use the combination of x and $\tilde{x} \triangleq \hat{x} - x$ as state. An equation for $\tilde{x}$ may be derived from Eqs. (III-1) and (III-6):

$$\dot{\tilde{x}} = f(\hat{x}, u, t) - f(x, u, t) + \hat{K}H(x - \hat{x}) + \hat{K}v - w + \frac{1}{2} f_{xx}^o o\hat{P}$$

$$\approx (f_x^o - \hat{K}H)\tilde{x} + \hat{K}v = w + \frac{1}{2} f_{xx}^o o(\hat{P} + \tilde{x}\tilde{x}^T)$$

$$E[\tilde{x}(0)] = 0 \qquad E[\tilde{x}(0)\tilde{x}^T(0)] = \hat{P}(0) \quad . \tag{B-7}$$

In terms of x and $\tilde{x}$ the Hamilton-Jacobi equation for the return function, $I(x, \tilde{x}, t)$ is*

$$0 = I_t + l + I_x f + I_{\tilde{x}} \left[(f_x^o - \hat{K}H)\tilde{x} + \frac{1}{2} f_{xx}^o o(\hat{P} + \tilde{x}\tilde{x}^T) \right]$$

$$+ \frac{1}{2} \operatorname{tr} [(I_{xx} - 2I_{x\tilde{x}} + I_{\tilde{x}\tilde{x}})\hat{Q} + I_{\tilde{x}\tilde{x}}\hat{K}\hat{R}\hat{K}^T]$$

$$I(x, \tilde{x}, T) = \Phi[x(T), T] \quad . \tag{B-8}$$

The second partials are characteristic of stochastic problems and do not appear in the deterministic problem. Performance is given by

$$J = E\{I(x, \tilde{x}, 0)\} \quad . \tag{B-9}$$

---

* No minimization appears because of the fact that the controller is fixed.

b. Approximate Solution of the *H-J* Equation by Perturbation Theory

We assume a solution to (B-8) that is quadratic in $\tilde{x}$ and the perturbation $x - x^o$:

$$I(x,\tilde{x},t) = J_d(t) + J_s(t) + \lambda^T(t)[x - x^o(t)] + \tilde{\lambda}^T\tilde{x}$$

$$+ [x - x^o(t)]^T P(t)[x - x^o(t)] + 2[x - x^o(t)]^T \overline{P}(t)\tilde{x} + \tilde{x}^T\tilde{P}(t)\tilde{x} \quad , \tag{B-10}$$

where J_d is the deterministic cost from time t:

$$J_d(t) = \int_t^T l(x^o,u^o,\tau)d\tau + \Phi[x^o(T),T] \quad . \tag{B-11}$$

The desired partials of I are

$$I_t = \dot{J}_d + \dot{J}_s - \lambda^T f^o + (\dot{\lambda}^T - 2f^{oT}P)(x - x^o) + (\dot{\tilde{\lambda}}^T - f^{oT}\overline{P})\tilde{x}$$

$$+ (x - x^o)^T\dot{P}(x - x^o) + 2(x - x^o)^T\dot{\overline{P}}x + \tilde{x}^T\dot{\tilde{P}}\tilde{x} \quad ,$$

$$I_x = \lambda^T + 2(x - x^o)P + 2\tilde{x}\overline{P}^T \quad ,$$

$$I_{\tilde{x}} = \tilde{\lambda}^T + 2(x - x^o)\overline{P} + 2\tilde{x}\tilde{P} \quad ,$$

$$I_{xx} = 2P \qquad I_{x\tilde{x}} = 2\overline{P} \qquad I_{\tilde{x}\tilde{x}} = 2\tilde{P} \quad . \tag{B-12}$$

Now let

$$H(x,u,\lambda,t) \triangleq l(x,u,t) + \lambda f(x,u,t) \tag{B-13}$$

and note that

$$u = u^o + K[(\hat{x} - x) + (x - x^o)] \quad . \tag{B-14}$$

then, by Taylor's series expansion

$$l + I_x f \approx H^\circ + (H_x^\circ - H_u^\circ K + f^{\circ T} P)(x - x^\circ) + (-H_u K + 2f^{\circ T}\overline{P})\tilde{x}$$

$$+ (x - x^\circ)^T \left[\frac{1}{2} H_{xx}^\circ - \frac{1}{2} H_{xu}^\circ K - \frac{1}{2} K^T H_{ux}^\circ + \frac{1}{2} K^T H_{uu}^\circ K + P(f_x^\circ - f_u^\circ K)\right.$$

$$\left. + (f_x^\circ - f_u^\circ K)P\right] (x - x^\circ)$$

$$+ 2(x - x^\circ)^T \left[-\frac{1}{2} H_{xu}^\circ K - \frac{1}{2} K H_{uu}^\circ K - P f_u^\circ K + (f_x^\circ - f_u^\circ K)^T \dot{\overline{P}}\right] \tilde{x}$$

$$+ \tilde{x}^T \left[-\frac{1}{2} K^T H_{uy}^\circ K + \overline{P}^T f_u^\circ K + (f_u^\circ K)^T \overline{P}\right] \tilde{x} \qquad . \qquad \text{(B-15)}$$

Use of (B-15) and (B-12) in (B-8) yields

$$0 = (\dot{J}_d + H^\circ - \lambda^T f^\circ) + \dot{J}_s + \mathrm{tr}\left[(P - 2\overline{P} + \hat{\tilde{P}})Q + \tilde{P}\hat{K}\hat{R}\hat{K}^T + \frac{1}{2}(\tilde{\lambda}^T f)_{xx}^\circ \hat{P}\right]$$

$$+ (\dot{\lambda}^T + H_x^\circ - H_u^\circ K)(x - x^\circ) + \left[\dot{\tilde{\lambda}} + \tilde{\lambda}(f_x^\circ - KH) - H_u^\circ K\right]\tilde{x} \qquad ,$$

$$+ (x - x^\circ)^T \left[\dot{P} + \frac{1}{2} H_{xx}^\circ - \frac{1}{2} H_{xu}^\circ K - \frac{1}{2} K^T H_{ux}^\circ + \frac{1}{2} K^T H_{uu}^\circ K + P(f_x^\circ + f_u^\circ K)\right.$$

$$\left. + (f_x^\circ - f_u^\circ K)P\right](x - x^\circ)$$

$$+ 2(x - x^\circ)^T \left[\dot{\overline{P}} - \frac{1}{2} H_{xu}^\circ K + \frac{1}{2} K^T H_{uu}^\circ K - P f^\circ K + (f_x^\circ - f_u^\circ K)^T \overline{P} + \overline{P}(f_x^\circ - \hat{K}H)\right]\tilde{x}$$

$$+ \tilde{x}^T \left[+\frac{1}{2} K^T H_{uu}^\circ K + \overline{P}^T f_u^\circ K + (f_u^\circ K)^T \overline{P} + \tilde{P}(f_x^\circ - \hat{K}H)\right.$$

$$\left. + (f_x^\circ - \hat{K}H) + (f_x^\circ - \hat{K}H)\tilde{P} + \frac{1}{2}(\tilde{\lambda}^T f)_{xx}^\circ\right]\tilde{x} \qquad . \qquad \text{(B-16)}$$

If it is remembered from Eqs. (B-11) and (B-13) that

$$\dot{J}_d = -l^\circ = -H^\circ + \lambda^T f^\circ \qquad , \qquad \text{(B-17)}$$

then the following equations may be written by equating the coefficients of various powers of $(x - x^{\circ})$ and $\tilde{x}$ in Eq. (B-16); final conditions are obtained by series expansion of Eq. (B-8) and comparison with Eq. (B-10):

$$-\dot{J}_s = \mathrm{tr}\left[(P - 2\overline{P} + \tilde{P})\hat{Q} + \tilde{P}\hat{K}\hat{R}\hat{K}^T + \frac{1}{2}(\tilde{\lambda}^T f)^{\circ}_{xx}\hat{P}\right] \qquad J_s(T) = 0 \tag{B-18}$$

$$-\dot{\lambda} = (H^{\circ}_x - H^{\circ}_u K)^T \qquad \lambda(T) = \Phi^{\circ}_x \tag{B-19}$$

$$-\dot{\tilde{\lambda}} = (f^{\circ}_x - \hat{K}H)^T\tilde{\lambda} - (H^{\circ}_u K)^T \qquad \tilde{\lambda}(T) = 0 \tag{B-20}$$

$$-\dot{P} = P(f^{\circ}_x - f^{\circ}_u K) + (f^{\circ}_x - f^{\circ}_u K)^T P + \frac{1}{2}H^{\circ}_{xx} - \frac{1}{2}H^{\circ}_{xu}K - \frac{1}{2}K^T H^{\circ}_{ux} + \frac{1}{2}K^T H^{\circ}_{uu}K$$

$$P(T) = \frac{1}{2}\Phi^{\circ}_{xx} \tag{B-21}$$

$$-\dot{\overline{P}} = \overline{P}(f^{\circ}_x - \hat{K}H) + (f^{\circ}_x - f^{\circ}_u K)^T\overline{P} - Pf^{\circ}_u K - \frac{1}{2}H^{\circ}_{xu}K + \frac{1}{2}K^T H^{\circ}_{uu}K$$

$$\overline{P}(T) = 0 \tag{B-22}$$

$$-\dot{\tilde{P}} = \tilde{P}(f^{\circ}_x - \hat{K}H) + (f^{\circ}_x - \hat{K}H)\tilde{P} + \overline{P}^T f^{\circ}_u K - (f^{\circ}_u K)^T\overline{P} + \frac{1}{2}K^T H^{\circ}_{uu}K$$

$$+ \frac{1}{2}(\tilde{\lambda}^T f)^{\circ}_{xx} \quad . \tag{B-23}$$

The performance J is given by [see Eq. (B-7) and Eq. (B-10)].

$$J = E[I(x,\hat{x},0)]$$

$$= J_d(t) + \lambda^T(0)[\hat{x}(0) - x^{\circ}(0)] + [\hat{x}(0) - x^{\circ}(0)]^T P[\hat{x}(0) - x^{\circ}(0)]$$

$$+ \mathrm{tr}\,[P\hat{P} + 2\overline{P}\hat{P} + 2\overline{P}\hat{P}] + \int_0^T \mathrm{tr}\left[(P - 2\overline{P} + \tilde{P})Q + \tilde{P}\hat{K}\hat{R}\hat{K}^T + \frac{1}{2}(\tilde{\lambda}^T f)^{\circ}_{xx}\hat{P}\right]dt \quad .$$

(B-24)

This equation may be reduced to the Eq. (III-11) for J given in the main text with the aid of Eqs. (B-21) and (B-22) and considerable algebra. Since this algebra is essentially the same as that given in Appendix C of Ref. 10, it will not be repeated here.

3. Program

Figure B-1 is a flow chart of a computer program utilizing the sensitivity and optimization theory presented in Sec. III. The equations solved in the various subroutines of the program are described in this portion of the appendix. Most of these equations are general for any system; some, however, vary greatly as a function of the system, and in these situations the equations given are those corresponding to the linear control problem:

$$\dot{x} = F(\alpha)x + G(\alpha)u \quad ,$$

$$J = \int_0^T (x^T Q x + u^T R u)dt + x^T(T)P_x(T)\,x(T) \quad , \qquad \text{(B-25)}$$

where F and G are linear functions of α, *i.e.*,

$$F^{(i,j)} = F^{o(i,j)} + \sum_{k=1}^{q} f_{\alpha x}^{(i,k,j)}\alpha^{(k)}$$

$$G^{(i,j)} = G^{o(i,j)} + \sum_{k=1}^{q} f_{\alpha u}^{(i,k,j)}\alpha^{(k)} \qquad \text{(B-26)}$$

for q parameters.

a. Mode Indicators

The mode of operation of program is determined by the following indicators, which are inputs.

No Measurement	$\gamma = 1$	$\varphi = 0$
Noisy Measurement	$\gamma = 1$	$\varphi = 1$
Perfect Measurement	$\gamma = 0$	$\varphi = 1$
Sensitivity	$\psi = 0$	
Optimization	$\psi = 1$	
Non Adaptive	$\xi = 0$	
Adaptive	$\xi = 1$	

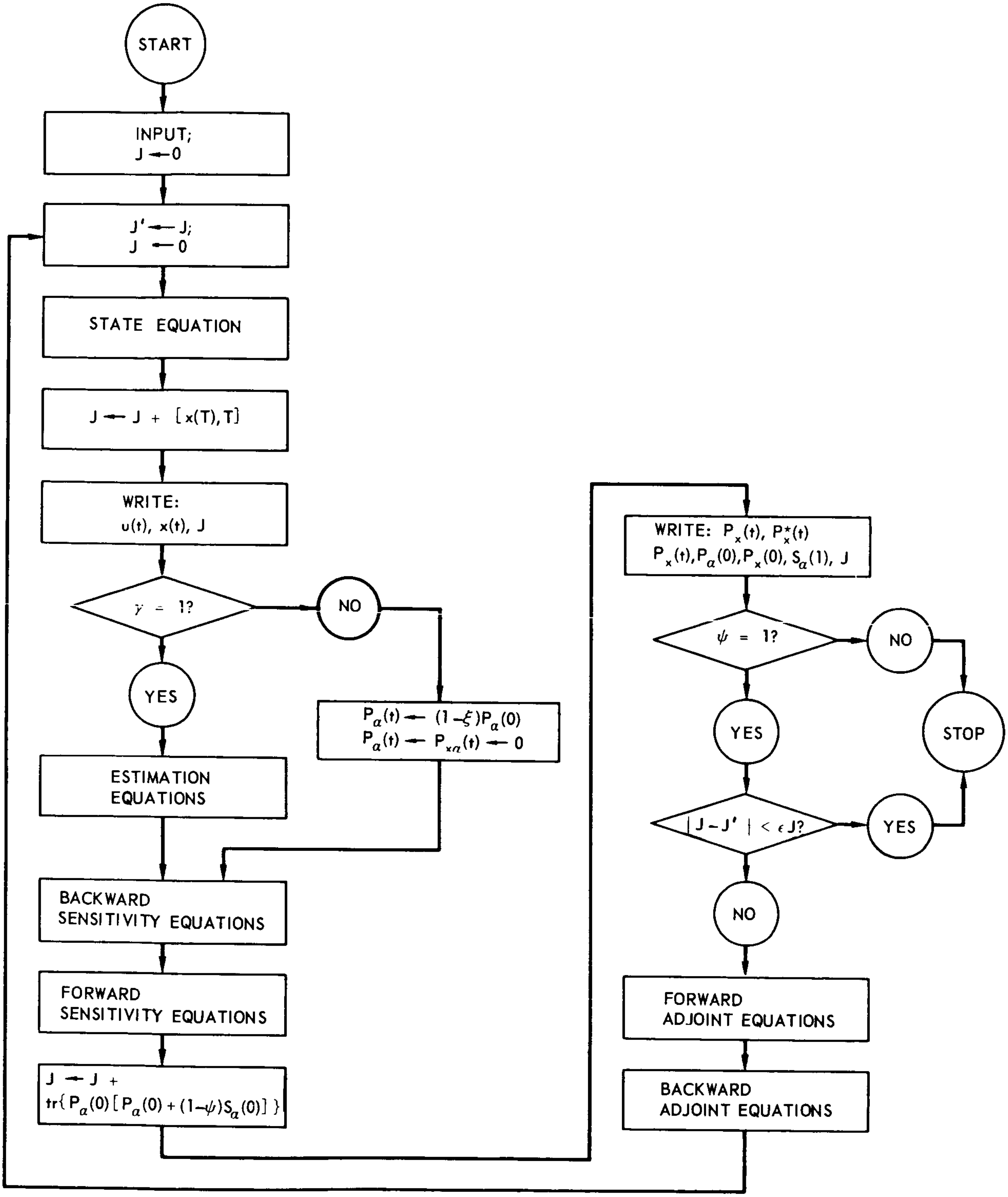

FIG. B-1 FLOW CHART FOR PROGRAM

b. Partial Derivatives

In the various equations partial derivatives of f and H appear and it is necessary to write subroutines to compute these derivatives. For the linear problem they are

$$f_x^o = F \quad , \quad f_u^o = G \quad , \quad H_x^o = 2x^{oT}Q + \lambda^T F \quad ,$$

$$H_u^o = 2u^{oT}R + \lambda^T G \quad , \quad H_\alpha^o = \lambda^T f_\alpha^o \quad , \quad H_{xx}^o = 2Q \quad ,$$

$$H_{\alpha\alpha}^o = H_{xu}^o = 0 \quad , \quad H_{uu}^o = 2R \quad , \quad \Phi_x^o(T) = 2x^{oT}(T)P_x(T) \quad ,$$

$$\Phi_{xx}^o(T) = 2P_x(T) \quad , \tag{B-27}$$

and

$$f_\alpha^{o(i,j)} = \sum_{k=1}^{n} F_{\alpha x}^{(i,j,k)} x^{(k)} + \sum_{k=1}^{m} F_{\alpha u}^{(i,j,k)} u^{(k)} \quad ,$$

$$H_{\alpha u}^{o(i,j)} = \sum_{k=1}^{n} \lambda^{(k)} F_{\alpha u}^{(k,i,j)} \quad ,$$

$$H_{\alpha x}^{o(i,j)} = \sum_{k=1}^{n} \lambda^{(k)} F_{\alpha x}^{(k,i,j)} \quad , \tag{B-28}$$

for n state variables and m inputs.

c. State Equation

$$\dot{x}^o = f(x^o, u^o, t) \quad ; \tag{B-29}$$

$u^o(t)$ and $x^o(0)$ are inputs.

d. Estimation Equation

$$\dot{\hat{P}}_x = \hat{Q} + F\hat{P}_x + \hat{P}_x F^T + \psi(F_\alpha \hat{P}_{x\alpha}^T + \hat{P}_{x\alpha} F_\alpha^T) - \varphi \hat{P}_x^* \quad , \tag{B-30}$$

$$\hat{P}_x^* = \hat{K} H P_x \quad , \quad \hat{K} = \hat{P}_x H^T \hat{R}^{-1}$$

except for $\psi = 1$ and $\xi = 0$ after first loop,

$$\hat{P}_x^* = \hat{K}H\hat{P}_x + \hat{P}_x H^T \hat{K} - \hat{K}\hat{R}\hat{K}^T$$

for $\psi = 1$ and $\xi = 0$ after first loop; $\hat{P}_x(0)$ is an input.

For $\psi = 1$ only:

$$\dot{\hat{P}}_{x\alpha} = F\hat{P}_{x\alpha} + F_\alpha \hat{P}_\alpha - \varphi \hat{K}H\hat{P}_{x\alpha} \quad , \quad \hat{P}_{x\alpha}(0) = 0 \quad . \tag{B-31}$$

$$\dot{\hat{P}}_\alpha = -\varphi \hat{K}_\alpha H\hat{P}_{x\alpha} \quad , \quad P_\alpha(0) \text{ is an input} \quad , \tag{B-32}$$

$$\hat{K}_\alpha^T = \xi \hat{R}^{-1} H\hat{P}_{x\alpha} \quad .$$

e. <u>Backward Sensitivity Equations</u>

$$-\dot{\lambda} = (H_x^\circ - H_u^\circ K)^T \quad , \quad \lambda(T) = \Phi_x^\circ(T) \quad . \tag{B-33}$$

$$-\dot{\lambda}_\alpha = H_\alpha^\circ \quad , \quad \lambda_\alpha(T) = 0 \quad . \tag{B-34}$$

$$-\dot{P}_x = \frac{1}{2} H_{xx}^\circ + f_x^{\circ T} P_x + P_x f_x^\circ - P_x^* \quad , \quad P_x(T) = \Phi_{xx}^\circ(T) \tag{B-35}$$

$$P_x^* = \left(P_x f_u^\circ + \frac{1}{2} H_{xu}^\circ\right)K \quad , \quad K = 2(H_{uu}^\circ)^{-1}\left(P_x f_u^\circ K + \frac{1}{2} H_{xu}^\circ\right) \quad ;$$

except that

$$P_x^* = \left(P_x f_u^\circ + \frac{1}{2} H_{xu}^\circ\right)K + K^T\left(P_x f_u^\circ K + \frac{1}{2} H_{xu}^\circ\right)^T - \frac{1}{2} K^T H_{uu}^\circ K \quad ,$$

for $\psi = 1$, $\xi = 0$, and $\varphi = 1$ after first loop.

$$-\dot{P}_{x\alpha} = \frac{1}{2} H_{x\alpha}^\circ + F^T P_{x\alpha} + P_x F_\alpha - P_{x\alpha}^* \quad , \quad P_{x\alpha}(T) = 0 \quad ,$$

$$P_{x\alpha}^* = \left(P_x f_u^\circ K + \frac{1}{2} H_{xu}^\circ\right)K_\alpha \quad ,$$

$$K_\alpha = 2(H_{uu}^\circ)^{-1}\left(P_{\alpha\alpha}^T f_u^\circ + \frac{1}{2} H_{\alpha u}^\circ\right)^T \quad . \tag{B-36}$$

$$-\dot{P}_\alpha = \frac{1}{2} H_{\alpha\alpha}^\circ + F_\alpha^T P_{x\alpha} + P_{x\alpha}^T F_\alpha - P_\alpha^* \quad , \quad P_\alpha(T) = 0 \quad ,$$

$$P_\alpha^* = \left(P_{x\alpha}^T f_u^\circ + \frac{1}{2} H_{\alpha u}^\circ\right)K_\alpha \quad . \tag{B-37}$$

For $\psi = 1$, $\xi = 0$ and $\varphi = 1$ after first loop only:

$$-\dot{\bar{P}}_x = (f_x^o - f_u^o K)^T \bar{P}_x + \bar{P}_x (f_x^o - \hat{K}H)$$

$$- \left(P_x f_u^o + \frac{1}{2} H_{xu}^o - \frac{1}{2} K^T H_{uu}^o\right) K \quad , \quad \bar{P}_x(T) = 0 \quad , \qquad \text{(B-38)}$$

$$-\dot{\bar{P}}_{x\alpha} = (f_x^o - f_u^o K)^T \bar{P}_{x\alpha} + \bar{P}_x f_\alpha^o$$

$$- \left(P_x f_u^o + \frac{1}{2} H_{xu}^o - \frac{1}{2} K^T H_{uu}^o\right) K_\alpha \quad , \quad \bar{P}_{x\alpha}(T) = 0 \quad . \qquad \text{(B-39)}$$

For all cases

$$-\dot{J} = \mathrm{tr}\,(P_x \hat{Q} + P_x^* \hat{P} + 2\psi A \bar{P}_x) + \psi\, \mathrm{tr}\,(2P_{x\alpha}^{*T} \hat{P}_{x\alpha} + P_\alpha^* \hat{P}_\alpha + 2B\bar{P}_{x\alpha}) \quad , \qquad \text{(B-40)}$$

$$A = (\hat{K}\hat{R} - \hat{P}_x H^T)\hat{K}^T \quad , \quad B = -\hat{P}_{x\alpha}^T H^T \hat{K}^T$$

for $\psi = 1$, $\xi = 0$, $\varphi = 1$ and $\gamma = 1$;

$$A = 0 \quad , \quad B = -\hat{P}_\alpha F_\alpha^T$$

for $\psi = 1$, $\xi = 0$, $\varphi = 1$ and $\gamma = 0$;

$$A = B = 0$$

otherwise.

f. Forward Sensitivity Equations

For $\gamma = 0$, $\xi = 0$, $\dot{S}_\alpha = P_\alpha^*$; for $\gamma = 0$, $\xi = 1$, $S_\alpha = 0$; for $\gamma = 1$

$$\dot{\theta}_{x\alpha} = (f_x^o - \varphi\hat{K}H)\theta_{x\alpha} + F_\alpha \theta_\alpha \quad , \quad \theta_{x\alpha}(0) = 0 \qquad \text{(B-41)}$$

$$\dot{\theta}_\alpha = -\varphi\psi\hat{K}_\alpha H \theta_{x\alpha} \qquad \theta_\alpha(0) = I \qquad \text{(B-42)}$$

$$\dot{S}_\alpha = \theta^T_{x\alpha}[P^*_x - H^T\hat{K}^T\overline{P}_x - (H^T\hat{K}^T\overline{P}_x)^T]\theta_{x\alpha}$$

$$+ \theta^T_{x\alpha}(P_{x\alpha} - H^T\hat{K}^T\overline{P}_{x\alpha})\theta_\alpha + \theta^T_\alpha(P_{x\alpha} - H^T\hat{K}^T\overline{P}_{x\alpha})^T\theta_{x\alpha}$$

$$+ \theta^T_\alpha P^*_\alpha\theta_\alpha \quad , \quad S_\alpha(0) = 0 \quad . \qquad \text{(B-43)}$$

g. Forward Adjoint Equations

$$\dot{\overline{x}} = -\frac{\partial \mathbf{H}}{\partial \lambda} \quad , \quad \overline{x}(0) = 0 \qquad \text{(B-44)}$$

where **H** is given in Eq. (III-34); for the linear problem this reduced to

$$\dot{\overline{x}} = (F - GK)\overline{x} + \overline{x}^*$$

$$\overline{x}^{*(i)} = \sum_{j=1}^{q}\sum_{k=1}^{n} [F^{(i.j.k)}_{\alpha x}\Gamma^{(j,k)}_{\alpha x} - F^{(i,j,k)}_{\alpha u}(K\Gamma^T_{\alpha x} + K_\alpha\Gamma_\alpha)^{(k,j)}] \qquad \text{(B-45)}$$

$$\dot{\Gamma}_x = -\hat{Q} + f^o_x\Gamma_x + \Gamma_x f^{oT}_x + f^o_\alpha\Gamma_{\alpha x} + \Gamma^T_{\alpha x}f^{oT}_\alpha$$

$$-f^o_uK(\Gamma_x + \hat{P}_x + \varphi(1-\xi)\overline{\Gamma}_x] - [\Gamma_x + \hat{P}_x + \varphi(1-\xi)\overline{\Gamma}_x]^T(f^o_uK)^T$$

$$-f^o_uK_\alpha[\Gamma_{\alpha x} + \hat{P}^T_{x\alpha} + \varphi(1-\xi)\overline{\Gamma}_{\alpha x}] - [\Gamma_{\alpha x} + \hat{P}^T_{x\alpha} + \varphi(1-\xi)\overline{\Gamma}_{\alpha x}]^T(f^o_uK_\alpha)^T \quad ,$$

$$\Gamma_x(0) = -\hat{P}_x(0) \quad . \qquad \text{(B-46)}$$

$$\dot{\Gamma}_{\alpha x} = \Gamma_{\alpha x}f^{oT}_x + \Gamma_\alpha f^{oT}_\alpha - (\Gamma_\alpha + \hat{P}_\alpha)(f^o_uK_\alpha)^T$$

$$-2[\Gamma_{\alpha x} + P_{\alpha x} + \varphi(1-\xi)\overline{\Gamma}_{\alpha x}P_xf^o_u(H^o_{uu})^{-1}]f^o_u + \varphi(1-\xi)\overline{\Gamma}_{\alpha x}K^Tf^o_u \quad ,$$

$$\Gamma_{\alpha x}(0) = 0 \quad . \qquad \text{(B-47)}$$

$$\dot{\Gamma}_\alpha = 0 \quad , \quad \Gamma_\alpha(0) = -\hat{P}_\alpha(0) \quad . \qquad \text{(B-48)}$$

For $\varphi = 1$, $\xi = 0$ only

$$\dot{\bar{\Gamma}} = \bar{\Gamma}_x(f_x^\circ - f_u K)^T + (f_x^\circ - \hat{KH})\Gamma_\alpha + F_\alpha \bar{\Gamma}_{\alpha x} - A \quad , \quad \Gamma_x(0) = 0 \quad . \quad (B\text{-}49)$$

$$\dot{\bar{\Gamma}}_{\alpha x} = \bar{\Gamma}_{\alpha x}(f_x^\circ - f_u^\circ K)^T - B \quad , \quad \Gamma_{\alpha x}(0) = 0 \quad . \quad (B\text{-}50)$$

To update u° it is necessary to compute H_u; for the linear case a portion of H_u involving the above quantities is

$$u' = + \frac{1}{2} K\bar{x} - R^{-1} u^* \quad , \quad (B\text{-}51)$$

$$u^{*(i)} = \sum_{j=1}^{n} \sum_{k=1}^{q} [P_x \Gamma_{\alpha x}^T + P_{x\alpha}\Gamma_\alpha + \varphi(1 - \xi)\bar{P}_x^T \bar{\Gamma}_{\alpha x}^T]^{(j,k)} F_{\alpha x}^{(j,k,i)} \quad . \quad (B\text{-}52)$$

For $\xi = 0$, $\varphi = 1$ a new value of K is computed by use of H_K; for the linear case this yields

$$K^T \leftarrow (1 - \det R)K^T + (\det R)\Big\{ P_x G + (\hat{P}_x + \Gamma_\alpha + \bar{\Gamma}_x + \bar{\Gamma}_x^T)^{-1} \Big[\bar{\Gamma}_x(\bar{P}_x - P_x)G$$

$$+ \bar{\Gamma}_{\alpha x}^T (\bar{P}_{x\alpha}^T G - K_\alpha^T R) + \frac{1}{2} \bar{x}(2u^{\circ T} R + \lambda^T G)\Big]\Big\} R^{-1} \quad . \quad (B\text{-}53)$$

h. Backward Adjoint Equations

For $\gamma = 1$, $\varphi = 1$ only:

$$-\dot{\hat{\Gamma}}_x = \hat{\Gamma}_x(f_x^\circ - \hat{KH}) + (f_x^\circ - \hat{KH})^T \hat{\Gamma}_x + P_x^*$$

$$-H^T K^T \hat{\bar{P}}_x - \bar{P}_x^T \hat{KH} \quad , \quad \hat{\Gamma}_x(T) = 0 \quad . \quad (B\text{-}54)$$

$$-\dot{\hat{\Gamma}}_{\alpha x} = F_\alpha^T \hat{\Gamma}_x + \Gamma_{\alpha x}(F - \hat{KH}) + P_{x\alpha}^{*T} - \bar{P}_{x\alpha}^T \hat{KH} \quad , \quad \hat{\Gamma}_{\alpha x}(T) = 0 \quad .$$

(B-55)

In computation of H_u and H_x the following terms arise in the linear problem

$$u^{*(i)} = \sum_{j=1}^{n} \sum_{k=1}^{q} (\hat{\Gamma}_x \hat{P}_{x\alpha} + \hat{\Gamma}_{\alpha x}^T \hat{P}_\alpha)^{(j,k)} F_{\alpha u}^{(j,k,i)}$$

$$\lambda^{*(i)} \leftarrow \lambda^{*(i)} - \sum_{j=1}^{n} \sum_{k=1}^{q} (\hat{\Gamma}_x \hat{P}_{x\alpha} + \hat{P}_{\alpha x}^T \hat{P}_\alpha)^{(j,k)} F_{\alpha x}^{(j,k,i)} \quad . \quad (B\text{-}56)$$

A new value of $\hat{K}$ is computed by use of $\mathbf{H}_{\hat{K}}$ for the linear case this yields

$$\hat{K} \leftarrow (1 - \det \hat{R})\hat{K} + (\det \hat{R})\{\hat{P}_x + (\Gamma_x + \overline{P}_x + \overline{P}_x^T)^{-1}$$

$$[\Gamma_{\alpha x}^T \hat{P}_{x\alpha}^T + \overline{P}_{x\alpha} \hat{P}_{x\alpha}^T - \overline{P}_x^T(\hat{P}_x + \overline{\Gamma}_x^T)]\}H^T\hat{R}^{-1} \quad . \qquad \text{(B-57)}$$

For all cases:

$$-\dot{\overline{\lambda}} = H_x \quad ; \quad \overline{\lambda}(T) = \Phi_x^o(T) \quad ; \qquad \text{(B-58)}$$

for the linear problem this reduces to

$$-\dot{\overline{\lambda}} = F^T\overline{\lambda} + 2Q(x - \overline{x}) + 2\lambda^{*(i)} \quad . \qquad \text{(B-59)}$$

A new value of u^o is found from $\mathbf{H}_u$; in the linear problem this value is

$$u^o \leftarrow (1 - \det R)u - (\det R)\left\{\frac{1}{2} R^{-1}[G^T\overline{\lambda} + \varphi\gamma u^*] + u\right\}$$

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APPENDIX C

PROOF OF SEPARATION OF PLANT AND MEASUREMENT CONTROL FOR LINEAR CASE

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APPENDIX C

PROOF OF SEPARATION OF PLANT AND MEASUREMENT CONTROL FOR LINEAR CASE

This appendix presents the proof that the plant control and measurement control can be optimized separately for the special case given in the text. Since the plant and measurement subsystem given by Eqs. (V-16) and (V-17) are linear in the state with additive Gaussian disturbances and measurement noise, the estimation described by Eq. (V-12) reduces to the Kalman filter, which is given in Eqs. (V-22) and (V-23). The derivation of the Kalman filter will not be repeated here; it is only necessary to note that

$$\mathfrak{p}_k = (\hat{x}_{k/k}, \hat{P}_{k/k}) \quad . \tag{C-1}$$

In the derivation the following lemma is needed:

$$E[x^T Q x] = \bar{x}^T Q \bar{x} + \mathrm{tr}\ [\bar{P}Q] \quad , \tag{C-2}$$

where

$$\bar{x} = E(x)$$

$$\bar{P} = E[(x - \bar{x})(x - \bar{x})^T]$$

and

$$\mathrm{tr}\ [AB] = \mathrm{tr}\ [BA] \quad . \tag{C-3}$$

This lemma may be easily proved by writing the matrix operations in terms of summations.

Use of Eqs. (V-16), and (V-18), and (C-2) yields

$$\underset{w_N}{E} \{x_{N+1}^T P_{N+1} x_{N+1}/x_N\} = (F_N x_N + G_N u_N^P)^T P_{N+1} \cdot (F_N x_N + G_N u_N^P) + \mathrm{tr}\ [P_{N+1}\hat{Q}_N] \quad . \tag{C-4}$$

Application of the lemma a second time and using Eqs. (V-14) and (V-19) gives

$$I_N(\mathfrak{p}_N, y_N) = \min_{u_N^P} [\hat{x}_{N/N} Q_N \hat{x}_{N/N} + u_N^T R_N u_N$$

$$+ (F_N \hat{x}_{N/N} + G_N u_N^P)^T P_{N+1} (F_N \hat{x}_{N/N} + G_N u_N^P)]$$

$$+ \mathrm{tr}\, [(F_N^T P_{N+1} F_N + Q_N) \hat{P}_{N/N} + P_{N+1} \hat{Q}_N]$$

$$= \hat{x}_{N/N}^T P_N \hat{x}_{N/N} + \mathrm{tr}\, [(P_{N+1}^* + P_N) \hat{P}_{N/N} - P_{N+1} \hat{Q}_N] \quad , \tag{C-5}$$

where P_{N+1}^* and P_N are given by Eq. (V-21), and the minimization is performed by completion of squares.

Substitution of Eq. (V-19) into Eq. (V-13) yields

$$L_k(\mathfrak{p}_k, u_k) = \hat{x}_{k/k}^T Q_k \hat{x}_{k/k} + u_k^{P^T} R_k u_k^P$$

$$+ l_k^M(u_k^M) + \mathrm{tr}\, [\hat{P}_{k/k} Q_k]$$

$$k = 0, \ldots, N-1 \quad . \tag{C-6}$$

Based upon the above form it is assumed that

$$I_{k+1}(\mathfrak{p}_{k+1}, y_{k+1}) = \hat{x}_{k+1/k+1}^T P_{k+1} \hat{x}_{k+1/k+1}$$

$$+ \mathrm{tr}\, [P_{k+1} \hat{P}_{k+1/k+1}]$$

$$+ I_{k+1}^M(y_{k+1}, \hat{P}_{k+1/k+1}) + b_{k+1} \quad , \tag{C-7}$$

where b_{k+1} is independent of u_k^P and u_k^M. From Eq. (C-1)

$$E\{\hat{x}_{k+1/k+1} / \hat{x}_{k/k}, \hat{P}_{k/k}\} = F_k \hat{x}_{k/k} + G_k u_k^P \quad . \tag{C-8}$$

Furthermore,

$$E\{[z_{k+1} - H_{k+1}(F_k\hat{x}_{k/k} + G_k u_k^P)]$$

$$\cdot\ [z_{k+1} - H_{k+1}(F_k\hat{x}_{k/k} + G_k u_k^P)]^T\}$$

$$= E\{[v_{k+1} + H_{k+1}(\hat{x}_{k+1} - F_k\hat{x}_{k/k} + G_k u_k^P)]$$

$$\cdot\ [v_{k+1} + H_{k+1}(x_{k+1} - F_k\hat{x}_{k/k} + G_k u_k^P)]^T\}$$

$$= R_{k+1} + H_{k+1}(F_k\hat{P}_{k/k}F_k^T + Q_k)H_{1+1}^T \quad . \tag{C-9}$$

Hence from Eqs. (V-23) and (C-2)

$$E\{\hat{x}_{k+1/k+1}^T P_{k+1}\hat{x}_{k+1/k+1}/\hat{x}_{k/k},\hat{P}_{k/k}\}$$

$$= (F_k\hat{x}_{k/k} + G_k u_k^P)^T P_{k+1}(F_k\hat{x}_{k/k} + G_k u_k^P)$$

$$+ \operatorname{tr}\ [P_{k+1}\hat{K}_{k+1}(R_{k+1} + H_{k+1}\hat{P}_{k+1/k}H_{k+1}^T)\hat{K}_{k+1}^T] \quad . \tag{C-10}$$

But from Eqs. (C-1), (V-23), (C-3), and (V-21),

$$\operatorname{tr}\ [P_{k+1}K_{k+1}(R_{k+1} + H_{k+1}\hat{P}_{k+1/k}H_{k+1}^T)\hat{K}_{k+1}^T]$$

$$= \operatorname{tr}\ [P_{k+1}\hat{P}_{k+1/k}H_{k+1}^T K_{k+1}^T]$$

$$= \operatorname{tr}\ [P_{k+1}(F_k\hat{P}_{k/k}F_k^T + Q_k - \hat{P}_{k+1/k+1})]$$

$$= \operatorname{tr}\ [(P_{k+1}^* + P_k - Q_k)\hat{P}_{k/k} + P_{k+1}(Q_k - \hat{P}_{k+1/k+1})] \quad . \tag{C-11}$$

Use of Eqs. (C-8), (C-10), and (C-11) in Eq. (V-14) results in

$$I_k(\mathfrak{p}_k, y_k) = \min_{u_k^P, u_{k+1}^M} \{\hat{x}_{k/k}^T Q_k \hat{x}_{k/k} + u_k^{P^T} R_k u_k^P$$

$$+ l_{k+1}^M(u_{k+1}^M) + \mathrm{tr}\,[\hat{P}_{k/k} Q_k]$$

$$+ (F_k \hat{x}_{k/k} + G_k u_k^P)^T P_{k+1} (F_k \hat{x}_{k/k} + G_k u_k^P)$$

$$+ \mathrm{tr}\,[(P_{k+1}^* + P_k - Q_k)\hat{P}_{k/k} + P_{k+1}(\hat{Q}_k - \hat{P}_{k+1/k+1})]$$

$$+ \mathrm{tr}\,[P_{k+1}\hat{P}_{k+1/k+1}] + I_{k+1}^M(y_{k+1}, \hat{P}_{k+1/k+1}) + b_{k+1}\}$$

$$= \min_{u_k^P} [\hat{x}_{k/k}^T Q_k \hat{x}_{k/k} + u_k^{P^T} R_k u_k^P + (F_k \hat{x}_{k/k} + G_k u_k^P)^T$$

$$\cdot P_{k+1}(F_k \hat{x}_{k/k} + G_k u_k^P)] + \mathrm{tr}\,[P_k \hat{P}_{k/k}]$$

$$+ \min_{u_{k+1}^M \in \mathcal{U}_{k+1}^M} \{l_k(u_k^M) + \mathrm{tr}\,[P_{k+1}^* \hat{P}_{k/k}]$$

$$+ I_{k+1}^M(y_{k+1}, \hat{P}_{k+1/k+1})\} + b_{k+1} + \mathrm{tr}\,[P_{k+1}\hat{Q}_k] \quad . \qquad \text{(C-12)}$$

The minimization over u_k^P can be performed by completion of squares to yield Eq. (V-21) for P_k. It is also seen from Eqs. (C-6) and (C-12) that if

$$b_k = \mathrm{tr}\,[P_{k+1}\hat{Q}_k] + b_{k+1} \qquad k = 0, \ldots, N-1$$

$$b_N = \mathrm{tr}\,[P_{N+1}\hat{Q}_N] \quad , \qquad \text{(C-13)}$$

then

$$I_k^M(y_k, \hat{P}_{k/k}) = \min_{u_{k+1}^M \in \mathcal{U}_{k+1}^M} \{l_{k+1}^M(u_{k+1}^M) + \mathrm{tr}\,[P_{k+1}^* \hat{P}_{k/k}]$$

$$+ I_{k+1}^M(y_{k+1}, \hat{P}_{k+1/k+1})\} \qquad k = 0, \ldots, N-1$$

$$I_N^M(y_N, \hat{P}_{N/N}) = \mathrm{tr}\,[P_{N+1}^* \hat{P}_{N/N}] \quad , \qquad \text{(C-14)}$$

where u_N^M must be chosen so $y_N \in \mathcal{Y}$.

Now from Eq. (C-15)

$$J = \min_{u_0^M \in \mathcal{U}_0^M} [I_0(\wp_0, y_0) + l_0^M(u_0^M)] = \bar{x}_0^T P_0 \bar{x}_0 + \mathrm{tr}\,[P_0 \hat{P}_{0/-1}]$$

$$+ b_0 + \min_{u_0^M \in \mathcal{U}_0^M} [I_0^M(y_0, \hat{P}_{0/0}) + l_0^M(u_0^M)] \quad . \tag{C-15}$$

Equation (C-14) and the last term of Eq. (C-15) are the dynamic programming equations for the nonlinear, deterministic control problem: Minimize

$$\sum_{k=0}^{N} \{ l_k^M(u_k^M) + \mathrm{tr}\,[P_{k+1}^* \hat{P}_{k/k}] \} \tag{C-16}$$

subject to the recursion equations for $\hat{P}_{k/k}$ and y_k, and the constraints

$$u_k^M \in \mathcal{U}_k^U \quad \text{and} \quad y_N \in \mathcal{Y} \quad .$$

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